LMSE 1

Logische Methoden des Software Engineerings

Vertiefungsmodul 1

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Diese Vorlesung

- Curry Howard Isomorphismus

- Lesen:
  – LCHI 4.1 – 4.3

- Übungen:
  – Folie 11, Folie 16

The Curry-Howard Isomorphism

• Curry, Haskell (1934), "Functionality in Combinatory Logic", *Proceedings of the National Academy of Sciences*, **20**, pp. 584-590.

• Curry, Haskell B.; Feys, Robert (1958), *Combinatory Logic Vol. I*, Amsterdam: North-Holland, with 2 sections by William Craig, see paragraph 9E

• Howard, William A. (09 1980) [original paper manuscript from 1969], "The formulae-as-types notion of construction", in ; Hindley, J. Roger, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Boston, MA: Academic Press, pp. 479–490, ISBN 978-0-12-349050-6.

The Curry-Howard Isomorphism

THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

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Dedicated to H. B. Curry on the occasion of his 80th birthday.

3. *Correspondence between derivation and terms*

THEOREM 1. Given any derivation of $\Gamma \rightarrow \beta$ in $P(\supset)$ we can find a construction of $\Gamma \rightarrow \beta$ and conversely.

Natural deduction and friends

- Proof systems:
  - Natural deduction
  - Sequent calculus
  - Hilbert-style systems
- Natural deduction
  - **G. Gentzen**: Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift* 39:176-210, 405-431, 1935
  - D. Prawitz: *Natural Deduction: A Proof Theoretical Study*. Almquist & Wiksell, 1965

Another form of N.D. proof trees

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi} \qquad \frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]^{(i)} \\ \vdots \\ \rho\end{array} \qquad \begin{array}{c}[\psi]^{(j)} \\ \vdots \\ \rho\end{array}}{\rho^{(i,j)}}$$

$$\frac{\begin{array}{c}[\varphi]^{(i)} \\ \vdots \\ \psi\end{array}}{\varphi \to \psi^{(i)}} \qquad\qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}$$

$$\frac{\bot}{\varphi}$$

Proof normalization

$$\dfrac{\dfrac{[\varphi]^{(1)}}{\varphi \to \varphi}(1) \qquad \dfrac{[\psi]^{(2)}}{\psi \to \psi}(2)}{\dfrac{(\varphi \to \varphi) \wedge (\psi \to \psi)}{\varphi \to \varphi}}$$

Detour:
I-rule followed by E-rule

$$\dfrac{[\varphi]^{(1)}}{\varphi \to \varphi}(1)$$

Detour
eliminated

Proof normalization

$$\dfrac{\dfrac{[\varphi]^{(3)}}{\varphi \longrightarrow \varphi^{(3)}} \qquad \dfrac{\dfrac{[\varphi \longrightarrow \varphi]^{(1)}}{\psi \longrightarrow \varphi \longrightarrow \varphi^{(2)}}}{(\varphi \longrightarrow \varphi) \longrightarrow \psi \longrightarrow \varphi \longrightarrow \varphi^{(1)}}}{\psi \longrightarrow \varphi \longrightarrow \varphi}$$

Unnecessary assumption (since it is provable)

$$\dfrac{\dfrac{[\varphi]^{(3)}}{\varphi \longrightarrow \varphi^{(3)}}}{\psi \longrightarrow \varphi \longrightarrow \varphi^{(2)}}$$

Unnecessary assumption eliminated (by proving it)

Proof normalization

$$\dfrac{\dfrac{\dfrac{\Sigma}{\varphi} \qquad \dfrac{\Pi}{\psi}}{\varphi \wedge \psi}}{\varphi} \quad \longrightarrow \quad \dfrac{\Sigma}{\varphi}$$

$$\dfrac{\dfrac{\Sigma}{\psi} \qquad \dfrac{\dfrac{\dfrac{[\psi]^{(i)}}{\Pi}}{\varphi}}{\psi \to \varphi^{(i)}}}{\varphi} \quad \longrightarrow \quad \dfrac{\dfrac{\dfrac{\Sigma}{\psi}}{\Pi}}{\varphi}$$

$$\dfrac{\dfrac{\dfrac{\Theta}{\varphi}}{\varphi \vee \psi} \qquad \dfrac{\dfrac{[\varphi]^{(i)}}{\Sigma}}{\rho} \qquad \dfrac{\dfrac{[\psi]^{(j)}}{\Pi}}{\rho}}{\rho^{(i,j)}} \quad \longrightarrow \quad \dfrac{\dfrac{\dfrac{\Theta}{\varphi}}{\Sigma}}{\rho}$$

The Curry-Howard Isomorphism

4.2.1. PROPOSITION (Curry-Howard isomorphism).

(i) *If* $\Gamma \vdash M : \varphi$ *then* $|\Gamma| \vdash \varphi$.

(ii) *If* $\Gamma \vdash \varphi$ *then there exists* $M \in \Lambda_{\Pi}$ *such that* $\Delta \vdash M : \varphi$, *where* $\Delta = \{(x_{\varphi} : \varphi) \mid \varphi \in \Gamma\}$.

PROOF. (i): by induction on the derivation of $\Gamma \vdash M : \varphi$.
(ii): by induction on the derivation of $\Gamma \vdash \varphi$. Let $\Delta = \{x_{\varphi} : \varphi \mid \varphi \in \Gamma\}$.

Übung

- Geben Sie einen vollständigen und detaillierten Beweis von Proposition 4.2.1.

4.2.2. Remark. The correspondence displays certain interesting problems with the natural deduction formulation of Chapter 2. For instance

$$\dfrac{\dfrac{x:\varphi, y:\varphi \vdash x:\varphi}{x:\varphi \vdash \lambda y{:}\varphi \,.\, x:\varphi\to\varphi}}{\vdash \lambda x{:}\varphi \,.\, \lambda y{:}\varphi \,.\, x:\varphi\to\varphi\to\varphi}$$

and

$$\dfrac{\dfrac{x:\varphi, y:\varphi \vdash y:\varphi}{x:\varphi \vdash \lambda y{:}\varphi \,.\, y:\varphi\to\varphi}}{\vdash \lambda x{:}\varphi \,.\, \lambda y{:}\varphi \,.\, y:\varphi\to\varphi\to\varphi}$$

are two different derivations in $\lambda\!\to$ showing that both $\lambda x{:}\varphi \,.\, \lambda y{:}\varphi \,.\, x$ and $\lambda x{:}\varphi \,.\, \lambda y{:}\varphi \,.\, y$ have type $\varphi\to\varphi\to\varphi$.

Both of these derivations are projected to

$$\dfrac{\dfrac{\varphi \vdash \varphi}{\varphi \vdash \varphi\to\varphi}}{\vdash \varphi\to\varphi\to\varphi}$$

This reflects the fact that, in the natural deduction system of Chapter 2, one cannot distinguish proofs in which assumptions are discharged in different orders. Indeed, $\lambda\!\to$ can be viewed as an extension of IPC($\to$) in which certain aspects such as this distinction are elaborated.

Extension w. product (conjunction) and disjoint sum (disjunction)

$$\begin{array}{lcll} \Lambda_\Pi & ::= & \ldots & \mid\ <\Lambda_\Pi, \Lambda_\Pi> \ \mid \pi_1(\Lambda_\Pi) \mid \pi_2(\Lambda_\Pi) \\ & & & \mid \mathrm{in}_1^{\psi \vee \varphi}(\Lambda_\Pi) \mid \mathrm{in}_2^{\psi \vee \varphi}(\Lambda_\Pi) \mid \mathrm{case}(\Lambda_\Pi; V.\Lambda_\Pi; V.\Lambda_\Pi) \end{array}$$

Typing rules (product, sum)

$$\frac{\Gamma \vdash M : \psi \quad \Gamma \vdash N : \varphi}{\Gamma \vdash < M, N >: \psi \wedge \varphi} \qquad \frac{\Gamma \vdash M : \psi \wedge \varphi}{\Gamma \vdash \pi_1(M) : \psi} \qquad \frac{\Gamma \vdash M : \psi \wedge \varphi}{\Gamma \vdash \pi_2(M) : \varphi}$$

$$\frac{\Gamma \vdash M : \psi}{\Gamma \vdash \mathrm{in}_1^{\psi \vee \varphi}(M) : \psi \vee \varphi} \qquad \frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \mathrm{in}_2^{\psi \vee \varphi}(M) : \psi \vee \varphi}$$

$$\frac{\Gamma \vdash L : \psi \vee \varphi \quad \Gamma, x : \psi \vdash M : \rho \quad \Gamma, y : \varphi \vdash N : \rho}{\Gamma \vdash \mathrm{case}(L; x.M; y.N) : \rho}$$

Reduction (product, sum)

$$
\begin{array}{lcl}
\pi_1(< M_1, M_2 >) & \rightarrow & M_1 \\
\pi_2(< M_1, M_2 >) & \rightarrow & M_2 \\
 & & \\
\mathrm{case}(\mathrm{in}_1^{\varphi}(N); x.K; y.L) & \rightarrow & K\{x := N\} \\
\mathrm{case}(\mathrm{in}_2^{\varphi}(N); x.K; y.L) & \rightarrow & L\{y := N\}
\end{array}
$$

Intuitively, $\phi \wedge \psi$ is a product type, so $< M_1, M_2 >$ is a pair, and $\pi_1(M)$ is the first projection. In type-free λ-calculus these could be defined in terms of pure λ-terms (see Proposition 1.46), but this is not possible in $\lambda\!\rightarrow$. This is related to the fact that one cannot define conjunction in IPC in terms of implication (contrary to the situation in classical logic, as we shall see later).

Übung

- Zeigen Sie, wie man mittels Disjunktionden boolschen Datentypen repräsentieren kann, so dass die Repräsentation semantisch korrekt ist (d.h. die Reduktion entspricht der Evaluierung eines IF-Statements).

Proof normalization is reduction!

- A **redex** is a *constructor* immediately surrounded by a corresponding *destructor*
  - An *abstraction* as operator in an *application*
  - A *projection* applied to a *pair*
  - A *case-switch* applied to an *injection*
- *When lambda-terms are viewed as proofs, reductions on terms are exactly proof normalization transformations*

Example

4.2.3. Example. Consider the following example deduction containing redundancy. The original derivation with constructions is:

$$\frac{x : \varphi \vdash x : \varphi}{\vdash \lambda x{:}\varphi \,.\, x : \varphi \to \varphi}$$

The complicated proof with constructions is:

$$\dfrac{\dfrac{\dfrac{y : \psi \vdash y : \psi}{\vdash \lambda y{:}\psi \,.\, y : \psi \to \psi} \qquad \dfrac{x : \varphi \vdash x : \varphi}{\vdash \lambda x{:}\varphi \,.\, x : \varphi \to \varphi}}{\vdash < \lambda x{:}\varphi \,.\, x, \lambda y{:}\psi \,.\, y >: (\varphi \to \varphi) \wedge (\psi \to \psi)}}{\vdash \pi_1(< \lambda x{:}\varphi \,.\, x, \lambda y{:}\psi \,.\, y >) : \varphi \to \varphi}$$

The construction of the latter proof tree in fact contains a redex which upon reduction yields the construction of the former proof tree.

BHK-interpretation by CHI

- *A construction of* $\varphi_1 \wedge \varphi_2$ *consists of a construction of* φ_1 *and a construction of* φ_2*;*
- *A construction of* $\varphi_1 \vee \varphi_2$ *consists of a number* $i \in \{1, 2\}$ *and a construction of of* φ_i*;*
- *A construction of* $\varphi_1 \rightarrow \varphi_2$ *is a method (function) transforming every construction of* φ_1 *into a construction of* φ_2*;*
- *There is no possible construction of* $\perp$ *(where* $\perp$ *denotes falsity).*

„Constructions“ are lambda-terms!
Propositions are types!
Proofs are type derivations!

CHI: summary

$\lambda_\rightarrow$	IPC($\rightarrow$)
term variable	assumption
term	construction (proof)
type variable	propositional variable
type	formula
type constructor	connective
inhabitation	provability
typable term	construction for a proposition
redex	construction representing proof tree with redundancy
reduction	normalization
value	normal construction