LMSE 1

Logische Methoden des Software Engineerings
Vertiefungsmodul 1

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Diese Vorlesung

- Church-style (explicit type system)
- Weak normalization

Lesen und Übungen

- Lesen: LNCH Kap. 3 (Rest)
- Übungen
  - Prove Proposition 3.1.8 (VL 4)
  - Prove Proposition 3.1.9 (VL 4)
  - Prove Corollary 3.1.11 (VL 4)

Simple types a la Church

3.2.1. Definition.

(i) The set Λ_Π of pseudo-terms is defined by the following grammar:

$$\Lambda_\Pi ::= V \mid (\lambda x{:}\Pi\ \Lambda_\Pi) \mid (\Lambda_\Pi\ \Lambda_\Pi)$$

where V is the set of (λ-term) variables and Π is the set of simple types.[1] We adopt the same terminology, notation, and conventions for pseudo-terms as for λ-terms, see 1.3–1.10, mutatis mutandis.

(ii) The *typability* relation $\Vdash$ on $C \times \Lambda_\Pi \times \Pi$ is defined by:[2]

$$\frac{}{\Gamma, x : \tau \vdash^* \ x : \tau} \qquad \frac{\Gamma, x : \sigma \vdash^* \ M : \tau}{\Gamma \vdash^* \ \lambda x{:}\sigma.M : \sigma \to \tau} \qquad \frac{\Gamma \vdash^* \ M : \sigma \to \tau \qquad \Gamma \vdash^* \ N : \sigma}{\Gamma \vdash^* \ M\ N : \tau}$$

where we require that $x \notin \mathrm{dom}(\Gamma)$ in the first and second rule.

(iii) The simply typed λ-calculus à la Church ($\lambda\!\to$ *à la Church*, for short) is the triple $(\Lambda_\Pi, \Pi, \Vdash)$.

(iv) If $\Gamma \vdash^* \ M : \sigma$ then we say that M *has type* σ *in* Γ. We say that $M \in \Lambda_\Pi$ is *typable* if there are Γ and σ such that $\Gamma \vdash^* \ M : \sigma$.

Example

3.2.2. Example. Let σ, τ, ρ be arbitrary simple types. Then:

(i) $\vdash^* \lambda x{:}\sigma.x : \sigma \to \sigma$;

(ii) $\vdash^* \lambda x{:}\sigma.\lambda y{:}\tau.x : \sigma \to \tau \to \sigma$;

(iii) $\vdash^* \lambda x{:}\sigma{\to}\tau{\to}\rho.\lambda y{:}\sigma{\to}\tau.\lambda z{:}\sigma.(x\ z)\ y\ z : (\sigma{\to}\tau{\to}\rho){\to}(\sigma{\to}\tau){\to}\sigma{\to}\rho$.

Special properties of Church-system

3.2.12. PROPOSITION (Uniqueness of types).

(i) *If* $\Gamma \vdash^* M : \sigma$ *and* $\Gamma \vdash^* M : \tau$ *then* $\sigma = \tau$.

(ii) *If* $\Gamma \vdash^* M : \sigma$ *and* $\Gamma \vdash^* N : \tau$ *and* $M =_\beta N$, *then* $\sigma = \tau$.

Weak normalization

Wir studieren den Beweis des folgenden Satzes, der erst von A.M. Turing skizziert wurde.

3.4.2. Theorem (Weak normalization). *Suppose* $\Gamma \vdash^* M : \sigma$. *Then there is a finite reduction* $M_1 \to_\beta M_2 \to_\beta \ldots \to_\beta M_n \in \mathrm{NF}_\beta$.

Height of a type

3.4.1. DEFINITION. Define the function $h : \Pi \to \mathbb{N}$ by:

$$\begin{array}{lcl} h(\alpha) & = & 0 \\ h(\tau \to \sigma) & = & 1 + \max(h(\tau), h(\sigma)) \end{array}$$

Weak normalization

3.4.2. THEOREM (Weak normalization). *Suppose* $\Gamma \vdash^* \ M : \sigma$. *Then there is a finite reduction* $M_1 \to_\beta M_2 \to_\beta \ldots \to_\beta M_n \in \mathrm{NF}_\beta$.

PROOF. We use a proof idea due independently to Turing and Prawitz.

Define the *height* of a redex $(\lambda x{:}\tau.P^\rho)R$ to be $h(\tau \to \rho)$. For $M \in \Lambda_\Pi$ with $M \notin \mathrm{NF}_\beta$ define

$$m(M) = (h(M), n)$$

where

$$h(M) = \max\{h(\Delta) \mid \Delta \text{ is a redex in } M\}$$

and n is the number of redex occurrences in M of height $h(M)$. If $M \in \mathrm{NF}_\beta$ we define $h(M) = (0, 0)$.

Normalization

We show by induction on lexicographically ordered pairs $m(M)$ that if M is typable in $\lambda\!\rightarrow$ à la Church, then M has a reduction to normal-form.

Let $\Gamma \vdash M : \sigma$. If $M \in \mathrm{NF}_\beta$ the assertion is trivially true. If $M \notin \mathrm{NF}_\beta$, let Δ be the rightmost redex in M of maximal height h (we determine the position of a subterm by the position of its leftmost symbol, i.e., the rightmost redex means the redex which *begins* as much to the right as possible).

Let M' be obtained from M by reducing the redex Δ. The term M' may in general have more redexes than M. But we claim that the number of redexes of height h in M' is smaller than in M. Indeed, the redex Δ has disappeared, and the reduction of Δ may only create new redexes of height less than h. To see this, note that the number of redexes can increase by either copying existing redexes or by creating new ones.

Normalization

Now observe that if a new redex is created then one of the following cases must hold:

1. The redex Δ is of the form $(\lambda x{:}\tau.\ \ldots xP^\rho\ \ldots)(\lambda y^\rho.Q^\mu)^\tau$, where $\tau = \rho \to \mu$, and reduces to $\ldots\ (\lambda y^\rho.Q^\mu)P^\rho\ \ldots$. There is a new redex $(\lambda y^\rho.Q^\mu)P^\rho$ of height $h(\tau) < h$.

2. We have $\Delta = (\lambda x{:}\tau.\lambda y{:}\rho.R^\mu)P^\tau$, occurring in the context $\Delta^{\rho\to\mu}Q^\rho$. The reduction of Δ to $\lambda y{:}\rho.R_1^\mu$, for some R_1, creates a new redex $(\lambda y{:}\rho.R_1^\mu)Q^\rho$ of height $h(\rho \to \mu) < h(\tau \to \rho \to \mu) = h$.

3. The last case is when $\Delta = (\lambda x{:}\tau.x)(\lambda y^\rho.P^\mu)$, with $\tau = \rho \to \mu$, and it occurs in the context $\Delta^\tau Q^\rho$. The reduction creates the new redex $(\lambda y^\rho.P^\mu)Q^\rho$ of height $h(\tau) < h$.

Normalization

The other possibility of adding redexes is by copying. If we have $\Delta = (\lambda x{:}\tau.P^{\rho})Q^{\tau}$, and P contains more than one free occurrence of x, then all redexes in Q are multiplied by the reduction. But we have chosen Δ to be the rightmost redex of height h, and thus all redexes in Q must be of smaller heights, because they are to the right of Δ.

Thus, in all cases $m(M) > m(M')$, so by the induction hypothesis M' has a normal-form, and then M also has a normal-form. $\square$

Expressibility

3.5.1. DEFINITION. Let

$$\mathbf{int} = (\alpha \to \alpha) \to (\alpha \to \alpha)$$

where α is an arbitrary type variable. A numeric function $f : \mathbb{N}^n \to \mathbb{N}$ is $\lambda{\to}$-*definable* if there is an $F \in \Lambda$ with $\vdash F : \mathbf{int} \to \cdots \to \mathbf{int} \to \mathbf{int}$ ($n+1$ occurrences of $\mathbf{int}$) such that

$$F\, c_{n_1} \ldots c_{n_m} =_\beta c_{f(n_1,\ldots,n_m)}$$

for all $n_1, \ldots, n_m \in \mathbb{N}$.

Expressibility

3.5.5. DEFINITION. The class of *extended polynomials* is the smallest class of numeric functions containing the

(i) *projections:* $U_i^m(n_1, \ldots, n_m) = n_i$ for all $1 \leq i \leq m$;

(ii) *constant functions:* $k(n) = k$;

(iii) *signum function:* $sg(0) = 0$ and $sg(m+1) = 1$.

and closed under *addition* and *multiplication:*

(i) *addition:* if $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^l \to \mathbb{N}$ are extended polynomials, then so is $(f+g) : \mathbb{N}^{k+l} \to \mathbb{N}$

$$(f+g)(n_1, \ldots, n_k, m_1, \ldots, m_l) = f(n_1, \ldots, n_k) + g(m_1, \ldots, m_l)$$

(ii) *multiplication:* if $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^l \to \mathbb{N}$ are extended polynomials, then so is $(f \cdot g) : \mathbb{N}^{k+l} \to \mathbb{N}$

$$(f \cdot g)(n_1, \ldots, n_k, m_1, \ldots, m_l) = f(n_1, \ldots, n_k) \cdot g(m_1, \ldots, m_l)$$

3.5.6. THEOREM (Schwichtenberg). *The $\lambda\!\to$-definable functions are exactly the extended polynomials.*