Logische Methoden des Software Engineerings

Combinators and combinatory logic

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Combinators

Let $\mathcal{C}$ be a set of *combinator symbols*, ranged over by X, Y, Z. The set $\Xi_{\mathcal{C}}$ of *combinatory expressions*, ranged over by F, G, H are defined inductively by:

- $X \in \mathcal{C} \Rightarrow X \in \Xi_{\mathcal{C}}$,
- $x \in \mathcal{V} \Rightarrow x \in \Xi_{\mathcal{C}}$,
- $F, G \in \Xi_{\mathcal{C}} \Rightarrow (FG) \in \Xi_{\mathcal{C}}$

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Consider the set SKI $= \{\mathbf{S}, \mathbf{K}, \mathbf{I}\}$ of combinator symbols (referred to as a combinatory *base*) and define the notion of *weak reduction* $\triangleright_w$ on Ξ_{SKI} by setting, for all $X, Y, Z \in \Xi_{\mathsf{SKI}}$:

$$\begin{array}{lll} \mathbf{I}F & \triangleright_w & F \\ \mathbf{K}FG & \triangleright_w & F \\ \mathbf{S}FGH & \triangleright_w & (FH)(GH) \end{array}$$

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Let $\rightarrow_w$ and $\twoheadrightarrow_w$ be the reduction relations on Ξ_{SKI} induced by $\triangleright_w$, by closing $\triangleright_w$ under contexts $C ::= [] \mid (CF) \mid (FC)$, $(F \in \Xi_{\mathsf{SKI}})$.

Combinatory bases

By choosing different sets $\mathfrak{B} \subseteq \mathcal{C}$ of combinators (such as $\mathfrak{B} = \mathsf{SKI} = \{\mathbf{S}, \mathbf{K}, \mathbf{I}\}$) we can study different combinatory calculi, since in each case we can consider a $\mathfrak{B}$-calculus generated from the combinators in $\mathfrak{B}$. In such cases, we refer to the set $\mathfrak{B}$ as a *combinatory base*.

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Exercise 1

Show that the combinator $\mathbf{I}$ *can be coded in terms of* $\mathbf{S}$ *and* $\mathbf{K}$*. Hint: Notice that* $(\mathbf{K}F)(\mathbf{K}F) \twoheadrightarrow_w F$*.*

In other words, the base SKI $= \{\mathbf{S}, \mathbf{K}, \mathbf{I}\}$ *is redundant. Or, in yet other words, the base SK* $= \{\mathbf{S}, \mathbf{K}\}$ *is complete with respect to SKI-calculus. For this reason, one also talks about SK-calculus.*

Combinatory bases

Schönfinkel used, in addition to $\mathbf{S}$ and $\mathbf{K}$, the combinators $\mathbf{B}$ and $\mathbf{C}$ with the definitions

$$
\begin{array}{lll}
\mathbf{B}FGH & \rhd_B & F(GH) \\
\mathbf{C}FGH & \rhd_C & FHG
\end{array}
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$$\begin{array}{lcl} \mathbf{B}FGH & \triangleright_B & F(GH) \\ \mathbf{C}FGH & \triangleright_C & FHG \end{array}$$

But they are not strictly needed, for we can take

$$\begin{array}{lcl} \mathbf{B} & \equiv & \mathbf{S}(\mathbf{KS})\mathbf{K} \\ \mathbf{C} & \equiv & \mathbf{S}(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})\mathbf{K}))\mathbf{S})(\mathbf{KK}) \end{array}$$

Combinatory bases

Exercise 2 (One point basis)

Define the combinator $\mathbf{X}$ *by the rule*

$$(\mathbf{X}F) \rhd_X ((F\mathbf{S})\mathbf{K})$$

Show that $(\mathbf{XX}) \twoheadrightarrow_X \mathbf{SK}(\mathbf{KK})$.
Show that $\mathbf{X}(\mathbf{X}(\mathbf{XX})) \twoheadrightarrow_X \mathbf{K}$.
Show that $\mathbf{X}(\mathbf{X}(\mathbf{X}(\mathbf{XX}))) \twoheadrightarrow_X \mathbf{S}$.
Conclude that $\{\mathbf{X}\}$ *is complete with respect to SKI-calculus.*

Define the map $(\_)_\Lambda : \Xi_{\mathsf{SKI}} \to \Lambda$ by induction on expressions in Ξ_{SKI}:

$$\begin{array}{lcl} (x)_\Lambda & \equiv & x,\ \text{for } x \in \mathcal{V} \\ (\mathbf{I})_\Lambda & \equiv & \lambda x.x \\ (\mathbf{K})_\Lambda & \equiv & \lambda xy.x \\ (\mathbf{S})_\Lambda & \equiv & \lambda xyz.(xz)(yz) \\ (FG)_\Lambda & \equiv & (F)_\Lambda (G)_\Lambda \end{array}$$

Proposition 1

If $F \twoheadrightarrow_w G$, then $(F)_\Lambda \twoheadrightarrow_\beta (G)_\Lambda$.

Exercise 3

Prove Proposition 1 by induction on the length of $F \twoheadrightarrow_w G$.

$\Lambda \mapsto \Xi_{\mathsf{SKI}}$

Define, for each $x \in \mathcal{V}$, the "bracket abstraction" map $[x] : \Xi_{\mathsf{SKI}} \to \Xi_{\mathsf{SKI}}$ by induction on expressions in Ξ_{SKI}:

$$
\begin{array}{lcl}
[x]x & \equiv & \mathbf{I} \\
[x]F & \equiv & \mathbf{K}F, \text{ if } x \notin \mathsf{FV}(F) \\
[x](FG) & \equiv & \mathbf{S}([x]F)([x]G), \text{ otherwise}
\end{array}
$$

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Proposition 2 (Combinatory completeness)

1. $\forall F \in \Xi_{\mathsf{SKI}}.\forall G \in \Xi_{\mathsf{SKI}}.\ ([x]F)G \twoheadrightarrow_w F[x := G]$
2. $\forall F \in \Xi_{\mathsf{SKI}}.\ ([x]F)_\Lambda \twoheadrightarrow_\beta \lambda x.(F)_\Lambda$
3. $\forall x \in \mathcal{V}.\ \forall F \in \Xi_{\mathsf{SKI}}.\ \exists H \in \Xi_{\mathsf{SKI}}.\ \forall G \in \Xi_{\mathsf{SKI}}.\ HG \twoheadrightarrow_w F[x := G]$

Exercise 4

Prove Proposition 2.

Define the map $(\_)_\Xi : \Lambda \to \Xi_{\mathsf{SKI}}$ by induction on λ-terms:

$$\begin{array}{lcl} (x)_\Xi & \equiv & x, \text{ for } x \in \mathcal{V} \\ (MN)_\Xi & \equiv & (M)_\Xi(N)_\Xi \\ (\lambda x.M)_\Xi & \equiv & [x](M)_\Xi \end{array}$$

Proposition 3

For all $M \in \Lambda$, one has $((M)_\Xi)_\Lambda \twoheadrightarrow_\beta M$.

Exercise 5

Prove Proposition 3.

Combinatory logic SKI

$$\frac{}{\Gamma, x : \tau \vdash_{\text{SKI}} x : \tau}\text{(var)}$$

$$\frac{}{\Gamma \vdash_{\text{SKI}} \mathbf{I} : \tau \to \tau}(\mathbf{I})$$

$$\frac{}{\Gamma \vdash_{\text{SKI}} \mathbf{K} : \tau \to \sigma \to \tau}(\mathbf{K})$$

$$\frac{}{\Gamma \vdash_{\text{SKI}} \mathbf{S} : (\tau \to \sigma \to \rho) \to (\tau \to \sigma) \to \tau \to \rho}(\mathbf{S})$$

$$\frac{\Gamma \vdash_{\text{SKI}} F : \tau \to \sigma \qquad \Gamma \vdash_{\text{SKI}} G : \tau}{\Gamma \vdash_{\text{SKI}} (FG) : \sigma}(\to\text{E})$$

Notice that variables x have fixed, *monomorphic types*, whereas combinators $\mathbf{S}, \mathbf{K}, \mathbf{I}$ have infinitely many types (their types are *schematic* or *polymorphic*).

Combinatory logic SKI

Lemma 1 (Deduction theorem for SKI)

If $\Gamma, x : \sigma \vdash_{\text{SKI}} F : \tau$*, then* $\Gamma \vdash_{\text{SKI}} [x]F : \sigma \to \tau$*.*

Proposition 4

1. *If* $\Gamma \vdash_{\text{SKI}} F : \tau$*, then* $\Gamma \vdash_{\Lambda} (F)_{\Lambda} : \tau$*.*
2. *If* $\Gamma \vdash_{\Lambda} M : \tau$*, then* $\Gamma \vdash_{\text{SKI}} (M)_{\Xi} : \tau$*.*

Exercise 6

Prove Proposition 4. Hint: The first statement is proven by induction on the derivation of $\Gamma \vdash_{\text{SKI}} F : \tau$*. The second statement is proven by induction on the derivation of* $\Gamma \vdash_{\Lambda} M : \tau$ *using Lemma 1.*