LMSE 1

Logische Methoden des Software Engineerings

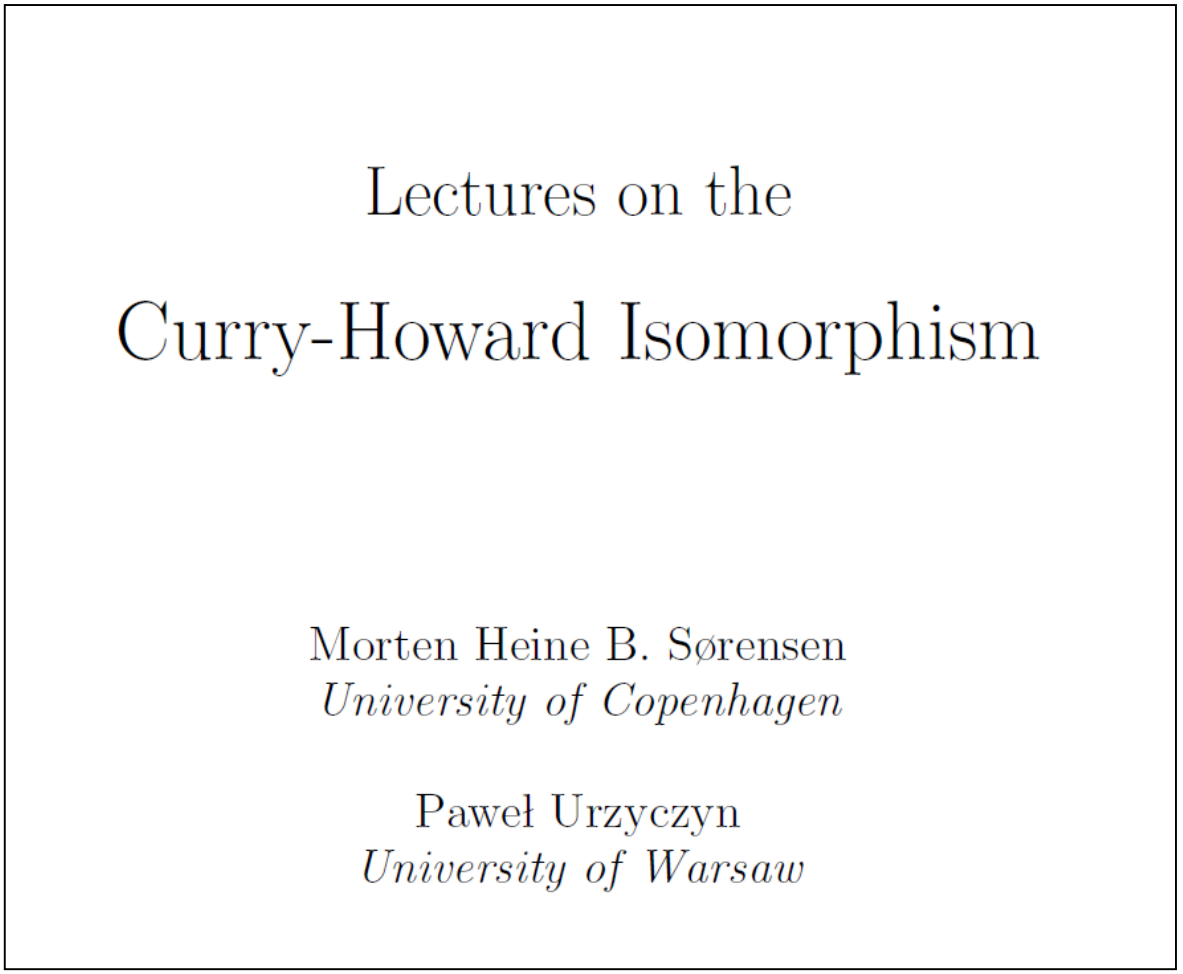
Vertiefungsmodul 1

Prof. Dr. Jakob Rehof

M.Sc. Andrej Dudenhefner

Lehrstuhl XIV, Software Engineering

Lecture Notes: LCHI



Lectures on the

Curry-Howard Isomorphism

Morten Heine B. Sørensen
University of Copenhagen

Paweł Urzyczyn
University of Warsaw

Datei: *curry-howard-notes.pdf*

LMSE 1 + 2:
Lambda Calculus and Type Theory

- *Lambda calculus* is one of the classical fundamental models of computation (Alonzo Church, around 1936), equivalent to Turing machines.
- *Lambda calculus* can be seen as a theoretical core of functional programming languages.
- *Typed lambda calculus* is a foundation for type systems in programming languages.
- *Typed lambda calculus* is the home and „lab“ for *type theory.*
- *Type theory* connects programming theory, category theory, and mathematical logic.
- *Higher type theory* is a foundation for (all of known) mathematics (e.g. *dependent type theory*).
- *Higher type theory* is a foundation for *theorem provers* (e.g. *Coq*)

Diese Vorlesung

- Definition des Lambda Kalküls
- Beta-Reduktion
- Church-Rosser Eigenschaft (Einleitung)

Lesen und Übungen

- Lesen: LCHI vom Anfang bis Abschn. 1.4 (S.8)

- Übungen
  - Exercise 1.7.4
  - Exercise 1.7.5
  - Exercise 1.7.6

λ-Calculus in One Slide

- λ-calculus is a minimalistic formalism for defining functional expressions and computations with such (a minimalistic functional programming language):
- Assuming computational expressions such as $x + 1$ the operation of λ-*abstraction* constructs the expression $\lambda x.x + 1$.
- An abstraction $\lambda x.M$ can be read as: "the function of x returning $M(x)$".
- Functional expressions are *anonymous*. Mathematicians sometimes use notation like $x \mapsto x + 1$, and that corresponds to $\lambda x.x + 1$.
- In addition to the operation of abstraction there is a (dual) operation of *application*: If M and N are λ-expressions, then the application of M to N, written $(M\ N)$, is also a λ-expression.
- Computation arises when abstractions are applied to arguments, for example $((\lambda x.x + 1)\ 2)$. An expression of the form $((\lambda x.M)\ N)$ is called a *redex*.
- Computation means *substitution* of arguments for the λ-abstracted variable of the function operator in a redex. For example:

$$((\lambda x.x + 1)\ 2) \rightarrow_\beta (x + 1)[x := 2] \equiv 2 + 1$$

Type free (untyped) lambda calculus

1.1.1. DEFINITION. Let

$$V = \{v_0, v_1, \ldots\}$$

denote an infinite alphabet. The set Λ^- of *pre-terms* is the set of strings defined by the grammar:

$$\Lambda^- ::= V \mid (\Lambda^- \ \Lambda^-) \mid (\lambda V \ \Lambda^-)$$

Pre-terms

1.1.2. EXAMPLE. The following are pre-terms.

(i) $((v_0\ v_1)\ v_2) \in \Lambda^-$;

(ii) $(\lambda v_0\ (v_0\ v_1)) \in \Lambda^-$;

(iii) $((\lambda v_0\ v_0)\ v_1) \in \Lambda^-$;

(iv) $((\lambda v_0\ (v_0\ v_0))\ (\lambda v_1\ (v_1\ v_1))) \in \Lambda^-$.

1.1.3. NOTATION. We use uppercase letters, e.g., K, L, M, N, P, Q, R with or without subscripts to denote arbitrary elements of Λ^- and lowercase letters, e.g., x, y, z with or without subscripts to denote arbitrary elements of V.

1.1.4. TERMINOLOGY.

(i) A pre-term of form x (i.e., an element of V) is called a *variable*;

(ii) A pre-term of form $(\lambda x\ M)$ is called an *abstraction* (over x);

(iii) A pre-term of form $(M\ N)$ is called an *application* (of M to N).

Notational conventions

1.1.5. NOTATION. We use the shorthands

(i) $(K\ L\ M)$ for $((K\ L)\ M)$;

(ii) $(\lambda x\ \lambda y\ M)$ for $(\lambda x\ (\lambda y\ M))$;

(iii) $(\lambda x\ M\ N)$ for $(\lambda x\ (M\ N))$;

(iv) $(M\ \lambda x\ N)$ for $(M\ (\lambda x\ N))$.

1.1.7. NOTATION. We write $\lambda x_1 \ldots x_n.M$ for $\lambda x_1\ \ldots \lambda x_n\ M$. As a special case, we write $\lambda x.M$ for $\lambda x\ M$.

Beispiel

1.1.9. EXAMPLE. The pre-terms in Example 1.1.2 can be written as follows, respectively:

(i) $v_0\ v_1\ v_2$;

(ii) $\lambda v_0.v_0\ v_1$;

(iii) $(\lambda v_0.v_0)\ v_1$;

(iv) $(\lambda v_0.v_0\ v_0)\ \lambda v_1.v_1\ v_1$.

1.1.10. REMARK. The conventions mentioned above are used in the remainder of these notes. However, we refrain from using them—wholly or partly—when we find this more convenient. For instance, we might prefer to write $(\lambda v_0.v_0\ v_0)\ (\lambda v_1.v_1\ v_1)$ for the last term in the above example.

Free variables, closed term

1.1.11. DEFINITION. For $M \in \Lambda^-$ define the set $\mathrm{FV}(M) \subseteq V$ of *free variables* of M as follows.

$$\begin{array}{lcl} \mathrm{FV}(x) & = & \{x\}; \\ \mathrm{FV}(\lambda x.P) & = & \mathrm{FV}(P)\backslash\{x\}; \\ \mathrm{FV}(P\,Q) & = & \mathrm{FV}(P) \cup \mathrm{FV}(Q). \end{array}$$

If $\mathrm{FV}(M) = \{\}$ then M is called *closed.*

Beispiel

1.1.12. EXAMPLE. Let x, y, z denote distinct variables. Then

(i) $\mathrm{FV}(x\ y\ z) = \{x, y, z\}$;

(ii) $\mathrm{FV}(\lambda x.x\ y) = \{y\}$;

(iii) $\mathrm{FV}((\lambda x.x\ x)\ \lambda y.y\ y) = \{\}$.

Substitution

1.1.13. DEFINITION. For $M, N \in \Lambda^-$ and $x \in V$, the *substitution of N for x in M*, written $M[x := N] \in \Lambda^-$, is defined as follows, where $x \neq y$:

$$
\begin{array}{lll}
x[x := N] & = N; & \\
y[x := N] & = y; & \\
(P\ Q)[x := N] & = P[x := N]\ Q[x := N]; & \\
(\lambda x.P)[x := N] & = \lambda x.P; & \\
(\lambda y.P)[x := N] & = \lambda y.P[x := N], & \text{if } y \notin \mathrm{FV}(N) \text{ or } x \notin \mathrm{FV}(P); \\
(\lambda y.P)[x := N] & = \lambda z.P[y := z][x := N], & \text{if } y \in \mathrm{FV}(N) \text{ and } x \in \mathrm{FV}(P).
\end{array}
$$

where z is chosen as the $v_i \in V$ with minimal i such that $v_i \notin \mathrm{FV}(P) \cup \mathrm{FV}(N)$ in the last clause.

Beispiel

1.1.14. EXAMPLE. If x, y, z are distinct variables, then for a certain variable u:

$$((\lambda x.x\ yz)\ (\lambda y.x\ y\ z)\ (\lambda z.x\ y\ z))[x := y] = (\lambda x.x\ yz)\ (\lambda u.y\ u\ z)\ (\lambda z.y\ y\ z)$$

Alpha-equivalence (alpha-conversion)

1.1.15. DEFINITION. Let *α-equivalence*, written $=_\alpha$, be the smallest relation on Λ^-, such that

$$\begin{array}{ll} P =_\alpha P & \text{for all } P; \\ \lambda x.P =_\alpha \lambda y.P[x := y] & \text{if } y \notin \mathrm{FV}(P), \end{array}$$

and closed under the rules:

$$\begin{array}{lcll} P =_\alpha P' & \Rightarrow & \forall x \in V: & \lambda x.P =_\alpha \lambda x.P'; \\ P =_\alpha P' & \Rightarrow & \forall Z \in \Lambda^-: & P\ Z =_\alpha P'\ Z; \\ P =_\alpha P' & \Rightarrow & \forall Z \in \Lambda^-: & Z\ P =_\alpha Z\ P'; \\ P =_\alpha P' & \Rightarrow & & P' =_\alpha P; \\ P =_\alpha P' \ \&\ P' =_\alpha P'' & \Rightarrow & & P =_\alpha P''. \end{array}$$

Beispiel

1.1.16. EXAMPLE. Let x, y, z denote different variables. Then

(i) $\lambda x.x =_\alpha \lambda y.y$;

(ii) $\lambda x.x\ z =_\alpha \lambda y.y\ z$;

(iii) $\lambda x.\lambda y.x\ y =_\alpha \lambda y.\lambda x.y\ x$;

(iv) $\lambda x.x\ y \neq_\alpha \lambda x.x\ z$.

Terms (modulo alpha-equivalence)

1.1.17. DEFINITION. Define for any $M \in \Lambda^-$, the *equivalence class* $[M]_\alpha$ by:

$$[M]_\alpha = \{N \in \Lambda^- \mid M =_\alpha N\}$$

Then define the set Λ of λ-*terms* by:

$$\Lambda \quad = \quad \Lambda^-/=_\alpha \quad = \quad \{[M]_\alpha \mid M \in \Lambda^-\}$$

1.1.19. NOTATION. We write M instead of $[M]_\alpha$ in the remainder. This leads to ambiguity: is M a pre-term or a λ-term? In the remainder of these notes, M should always be construed as $[M]_\alpha \in \Lambda$, *except when explicitly stated otherwise.*

Free variables (modulo alpha)

1.1.20. DEFINITION. For $M \in \Lambda$ define the set $\mathrm{FV}(M) \subseteq V$ of *free variables* of M as follows.

$$\begin{array}{lcl} \mathrm{FV}(x) & = & \{x\}; \\ \mathrm{FV}(\lambda x.P) & = & \mathrm{FV}(P)\backslash\{x\}; \\ \mathrm{FV}(P\ Q) & = & \mathrm{FV}(P) \cup \mathrm{FV}(Q). \end{array}$$

If $\mathrm{FV}(M) = \{\}$ then M is called *closed.*

1.1.21. REMARK. According to Notation 1.1.19, what we really mean by this is that we define FV as the map from Λ to subsets of V satisfying the rules:

$$\begin{array}{lcl} \mathrm{FV}([x]_\alpha) & = & \{x\}; \\ \mathrm{FV}([\lambda x.P]_\alpha) & = & \mathrm{FV}([P]_\alpha)\backslash\{x\}; \\ \mathrm{FV}([P\ Q]_\alpha) & = & \mathrm{FV}([P]_\alpha) \cup \mathrm{FV}([Q]_\alpha). \end{array}$$

Substitution (modulo alpha)

1.1.22. DEFINITION. For $M, N \in \Lambda$ and $x \in V$, the *substitution of N for x in M*, written $M\{x := N\}$, is defined as follows:

$$
\begin{array}{lcll}
x[x := N] & = & N; & \\
y[x := N] & = & y, & \text{if } x \neq y; \\
(P\ Q)[x := N] & = & P[x := N]\ Q[x := N]; & \\
(\lambda y.P)[x := N] & = & \lambda y.P[x := N], & \text{if } x \neq y, \text{ where } y \notin \mathrm{FV}(N).
\end{array}
$$

1.1.23. EXAMPLE.

(i) $(\lambda x.x\ y)[x := \lambda z.z] = \lambda x.x\ y$;

(ii) $(\lambda x.x\ y)[y := \lambda z.z] = \lambda x.x\ \lambda z.z$.

Beta notion of reduction

1.2.1. DEFINITION. Let $\rightarrow_\beta$ be the smallest relation on Λ such that

$$(\lambda x.P)\ Q \rightarrow_\beta P[x := Q],$$

and closed under the rules:

$$\begin{array}{lcll} P \rightarrow_\beta P' & \Rightarrow & \forall x \in V: & \lambda x.P \rightarrow_\beta \lambda x.P' \\ P \rightarrow_\beta P' & \Rightarrow & \forall Z \in \Lambda: & P\ Z \rightarrow_\beta P'\ Z \\ P \rightarrow_\beta P' & \Rightarrow & \forall Z \in \Lambda: & Z\ P \rightarrow_\beta Z\ P' \end{array}$$

A term of form $(\lambda x.P)\ Q$ is called a *β-redex,* and $P[x := Q]$ is called its *β-contractum.* A term M is a *β-normal form* if there is no term N with $M \rightarrow_\beta N$.

Beta-reduction and conversion

1.2.2. DEFINITION.

(i) The relation $\twoheadrightarrow_\beta$ (*multi-step β-reduction*) is the transitive-reflexive closure of $\rightarrow_\beta$; that is, $\twoheadrightarrow_\beta$ is the smallest relation closed under the rules:

$$\begin{array}{lll} P \rightarrow_\beta P' & \Rightarrow & P \twoheadrightarrow_\beta P'; \\ P \twoheadrightarrow_\beta P' \ \& \ P' \twoheadrightarrow_\beta P'' & \Rightarrow & P \twoheadrightarrow_\beta P''; \\ P \twoheadrightarrow_\beta P. & & \end{array}$$

(ii) The relation $=_\beta$ (*β-equality*) is the transitive-reflexive-symmetric closure of $\rightarrow_\beta$; that is, $=_\beta$ is the smallest relation closed under the rules:

$$\begin{array}{lll} P \rightarrow_\beta P' & \Rightarrow & P =_\beta P'; \\ P =_\beta P' \ \& \ P' =_\beta P'' & \Rightarrow & P =_\beta P''; \\ P =_\beta P; & & \\ P =_\beta P' & \Rightarrow & P' =_\beta P. \end{array}$$

Warning (many equalities!)

1.2.3. WARNING. In these notes, the symbol $=$ without any qualification is used to express the fact that two objects, e.g., pre-terms or λ-terms are identical. This symbol is very often used in the literature for β-equality.

Beispiel

1.2.4. EXAMPLE.

(i) $(\lambda x.x\ x)\ \lambda z.z \to_\beta (x\ x)[x := \lambda z.z] = (\lambda z.z)\ \lambda y.y;$

(ii) $(\lambda z.z)\ \lambda y.y \to_\beta z[z := \lambda y.y] = \lambda y.y;$

(iii) $(\lambda x.x\ x)\ \lambda z.z \twoheadrightarrow_\beta \lambda y.y;$

(iv) $(\lambda x.x)\ y\ z =_\beta y\ ((\lambda x.x)\ z).$