

LMSE

Logische Methoden des Software
Engineerings

Prof. Dr. Jakob Rehof
Lehrstuhl XIV, Software
Engineering

Diese Vorlesung

- Curry Howard Isomorphismus
- Lesen:
 - LCHI 4.1 – 4.3
- Übungen:
 - Folie 11, Folie 16

The Curry-Howard Isomorphism



- Curry, Haskell (1934), "Functionality in Combinatory Logic", *Proceedings of the National Academy of Sciences*, **20**, pp. 584-590.
- Curry, Haskell B.; Feys, Robert (1958), *Combinatory Logic Vol. I*, Amsterdam: North-Holland, with 2 sections by William Craig, see paragraph 9E
- Howard, William A. (09 1980) [original paper manuscript from 1969], "The formulae-as-types notion of construction", in ; [Hindley, J. Roger](#), *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Boston, MA: [Academic Press](#), pp. 479–490, [ISBN 978-0-12-349050-6](#).

The Curry-Howard Isomorphism

THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

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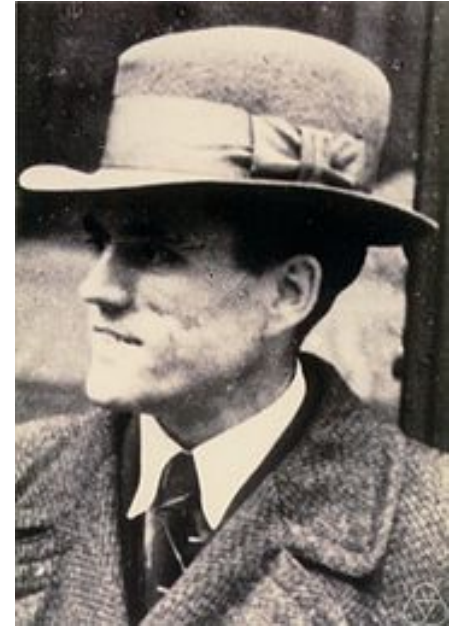
Dedicated to H. B. Curry on the occasion of his 80th birthday.

3. Correspondence between derivation and terms

THEOREM 1. Given any derivation of $\Gamma \rightarrow \beta$ in $P(\supset)$ we can find a construction of $\Gamma \rightarrow \beta$ and conversely.

Natural deduction and friends

- Proof systems:
 - Natural deduction
 - Sequent calculus
 - Hilbert-style systems
- Natural deduction
 - **G. Gentzen**: Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift* 39:176-210, 405-431, 1935
 - D. Prawitz: *Natural Deduction: A Proof Theoretical Study*. Almquist & Wiksell, 1965



Another form of N.D. proof trees

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

$$\frac{\varphi}{\varphi \vee \psi}$$

$$\frac{\psi}{\varphi \vee \psi}$$

$$\frac{\varphi \vee \psi \quad \begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \rho \end{array} \quad \begin{array}{c} [\psi]^{(j)} \\ \vdots \\ \rho \end{array}}{\rho^{(i,j)}}$$

$$\frac{\begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi^{(i)}}$$

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi}$$

$$\frac{\perp}{\varphi}$$

Proof normalization

$$\frac{[\varphi]^{(1)}}{\varphi \rightarrow \varphi^{(1)}} \quad \frac{[\psi]^{(2)}}{\psi \rightarrow \psi^{(2)}}$$

$$\frac{(\varphi \rightarrow \varphi) \wedge (\psi \rightarrow \psi)}{\varphi \rightarrow \varphi}$$

Detour:
I-rule followed by E-rule

$$\frac{[\varphi]^{(1)}}{\varphi \rightarrow \varphi^{(1)}}$$

Detour
eliminated

Proof normalization

$$\frac{\frac{[\varphi]^{(3)}}{\varphi \rightarrow \varphi^{(3)}} \quad \frac{\frac{[\varphi \rightarrow \varphi]^{(1)}}{\psi \rightarrow \varphi \rightarrow \varphi^{(2)}}}{(\varphi \rightarrow \varphi) \rightarrow \psi \rightarrow \varphi \rightarrow \varphi^{(1)}}}{\psi \rightarrow \varphi \rightarrow \varphi}$$

Unnecessary assumption
(since it is provable)

$$\frac{\frac{[\varphi]^{(3)}}{\varphi \rightarrow \varphi^{(3)}}}{\psi \rightarrow \varphi \rightarrow \varphi^{(2)}}$$

Unnecessary assumption
eliminated (by proving it)

Proof normalization

$$\frac{\frac{\Sigma}{\varphi} \quad \frac{\Pi}{\psi}}{\frac{\varphi \wedge \psi}{\varphi}} \rightarrow \frac{\Sigma}{\varphi}$$

$$\frac{\frac{\Sigma}{\psi} \quad \frac{\frac{[\psi]^{(i)}}{\Pi}}{\varphi}}{\psi \rightarrow \varphi^{(i)}}}{\varphi} \rightarrow \frac{\frac{\Sigma}{\psi}}{\Pi} \varphi$$

$$\frac{\frac{\Theta}{\varphi} \quad \frac{\frac{[\varphi]^{(i)}}{\Sigma}}{\rho} \quad \frac{\frac{[\psi]^{(j)}}{\Pi}}{\rho}}{\varphi \vee \psi}}{\rho^{(i,j)}} \rightarrow \frac{\frac{\Theta}{\varphi}}{\Sigma} \rho$$

The Curry-Howard Isomorphism

4.2.1. PROPOSITION (Curry-Howard isomorphism).

- (i) *If $\Gamma \vdash M : \varphi$ then $|\Gamma| \vdash \varphi$.*
- (ii) *If $\Gamma \vdash \varphi$ then there exists $M \in \Lambda_{\Pi}$ such that $\Delta \vdash M : \varphi$, where $\Delta = \{(x_{\varphi} : \varphi) \mid \varphi \in \Gamma\}$.*

PROOF. (i): by induction on the derivation of $\Gamma \vdash M : \varphi$.

(ii): by induction on the derivation of $\Gamma \vdash \varphi$. Let $\Delta = \{x_{\varphi} : \varphi \mid \varphi \in \Gamma\}$.

Übung

- Geben Sie einen vollständigen und detaillierten Beweis von Proposition 4.2.1.

4.2.2. REMARK. The correspondence displays certain interesting problems with the natural deduction formulation of Chapter 2. For instance

$$\frac{\frac{x : \varphi, y : \varphi \vdash x : \varphi}{x : \varphi \vdash \lambda y : \varphi . x : \varphi \rightarrow \varphi}}{\vdash \lambda x : \varphi . \lambda y : \varphi . x : \varphi \rightarrow \varphi \rightarrow \varphi}$$

and

$$\frac{\frac{x : \varphi, y : \varphi \vdash y : \varphi}{x : \varphi \vdash \lambda y : \varphi . y : \varphi \rightarrow \varphi}}{\vdash \lambda x : \varphi . \lambda y : \varphi . y : \varphi \rightarrow \varphi \rightarrow \varphi}$$

are two different derivations in $\lambda \rightarrow$ showing that both $\lambda x : \varphi . \lambda y : \varphi . x$ and $\lambda x : \varphi . \lambda y : \varphi . y$ have type $\varphi \rightarrow \varphi \rightarrow \varphi$.

Both of these derivations are projected to

$$\frac{\frac{\varphi \vdash \varphi}{\varphi \vdash \varphi \rightarrow \varphi}}{\vdash \varphi \rightarrow \varphi \rightarrow \varphi}$$

This reflects the fact that, in the natural deduction system of Chapter 2, one cannot distinguish proofs in which assumptions are discharged in different orders. Indeed, $\lambda \rightarrow$ can be viewed as an extension of $\text{IPC}(\rightarrow)$ in which certain aspects such as this distinction are elaborated.

Extension w. product (conjunction) and disjoint sum (disjunction)

$$\Lambda_{\Pi} ::= \dots \mid \langle \Lambda_{\Pi}, \Lambda_{\Pi} \rangle \mid \pi_1(\Lambda_{\Pi}) \mid \pi_2(\Lambda_{\Pi}) \\ \mid \text{in}_1^{\psi \vee \varphi}(\Lambda_{\Pi}) \mid \text{in}_2^{\psi \vee \varphi}(\Lambda_{\Pi}) \mid \text{case}(\Lambda_{\Pi}; V.\Lambda_{\Pi}; V.\Lambda_{\Pi})$$

Typing rules (product, sum)

$$\frac{\Gamma \vdash M : \psi \quad \Gamma \vdash N : \varphi}{\Gamma \vdash \langle M, N \rangle : \psi \wedge \varphi}$$

$$\frac{\Gamma \vdash M : \psi \wedge \varphi}{\Gamma \vdash \pi_1(M) : \psi}$$

$$\frac{\Gamma \vdash M : \psi \wedge \varphi}{\Gamma \vdash \pi_2(M) : \varphi}$$

$$\frac{\Gamma \vdash M : \psi}{\Gamma \vdash \text{in}_1^{\psi \vee \varphi}(M) : \psi \vee \varphi}$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \text{in}_2^{\psi \vee \varphi}(M) : \psi \vee \varphi}$$

$$\frac{\Gamma \vdash L : \psi \vee \varphi \quad \Gamma, x : \psi \vdash M : \rho \quad \Gamma, y : \varphi \vdash N : \rho}{\Gamma \vdash \text{case}(L; x.M; y.N) : \rho}$$

Reduction (product, sum)

$$\pi_1(\langle M_1, M_2 \rangle) \rightarrow M_1$$

$$\pi_2(\langle M_1, M_2 \rangle) \rightarrow M_2$$

$$\text{case}(\text{in}_1^\varphi(N); x.K; y.L) \rightarrow K\{x := N\}$$

$$\text{case}(\text{in}_2^\varphi(N); x.K; y.L) \rightarrow L\{y := N\}$$

Intuitively, $\phi \wedge \psi$ is a product type, so $\langle M_1, M_2 \rangle$ is a pair, and $\pi_1(M)$ is the first projection. In type-free λ -calculus these could be defined in terms of pure λ -terms (see Proposition 1.46), but this is not possible in $\lambda \rightarrow$. This is related to the fact that one cannot define conjunction in IPC in terms of implication (contrary to the situation in classical logic, as we shall see later).

Übung

- Zeigen Sie, wie man mittels Disjunktionen booleschen Datentypen repräsentieren kann, so dass die Repräsentation semantisch korrekt ist (d.h. die Reduktion entspricht der Evaluierung eines IF-Statements).

Proof normalization is reduction!

- A **redex** is a *constructor* immediately surrounded by a corresponding *destructor*
 - An *abstraction* as operator in an *application*
 - A *projection* applied to a *pair*
 - A *case-switch* applied to an *injection*
- *When lambda-terms are viewed as proofs, reductions on terms are exactly proof normalization transformations*

Example

4.2.3. EXAMPLE. Consider the following example deduction containing redundancy. The original derivation with constructions is:

$$\frac{x : \varphi \vdash x : \varphi}{\vdash \lambda x : \varphi . x : \varphi \rightarrow \varphi}$$

The complicated proof with constructions is:

$$\frac{\frac{y : \psi \vdash y : \psi}{\vdash \lambda y : \psi . y : \psi \rightarrow \psi} \quad \frac{x : \varphi \vdash x : \varphi}{\vdash \lambda x : \varphi . x : \varphi \rightarrow \varphi}}{\vdash \langle \lambda x : \varphi . x, \lambda y : \psi . y \rangle : (\varphi \rightarrow \varphi) \wedge (\psi \rightarrow \psi)}}{\vdash \pi_1(\langle \lambda x : \varphi . x, \lambda y : \psi . y \rangle) : \varphi \rightarrow \varphi}$$

The construction of the latter proof tree in fact contains a redex which upon reduction yields the construction of the former proof tree.

BHK-interpretation by CHI

- *A construction of $\varphi_1 \wedge \varphi_2$ consists of a construction of φ_1 and a construction of φ_2 ;*
- *A construction of $\varphi_1 \vee \varphi_2$ consists of a number $i \in \{1, 2\}$ and a construction of φ_i ;*
- *A construction of $\varphi_1 \rightarrow \varphi_2$ is a method (function) transforming every construction of φ_1 into a construction of φ_2 ;*
- *There is no possible construction of \perp (where \perp denotes falsity).*

„Constructions“ are lambda-terms!
Propositions are types!
Proofs are type derivations!

CHI: summary

$\lambda \rightarrow$

term variable

term

type variable

type

type constructor

inhabitation

typable term

redex

reduction

value

IPC(\rightarrow)

assumption

construction (proof)

propositional variable

formula

connective

provability

construction for a proposition

construction representing proof tree with redundancy

normalization

normal construction