

# LMSE

Logische Methoden des Software  
Engineerings

Prof. Dr. Jakob Rehof  
Lehrstuhl XIV, Software  
Engineering

# Diese Vorlesung

- Intuitionistische Logik
- Natürliche Deduktion

# Lesen und Übungen

- Lesen Kapitel 2.1 und 2.2 (Seite 23 - 28)
- Übungen
  - Aufgabe 2.7.3
  - Zeigen Sie mittels natürlicher Deduktion, dass die Formeln

$$\varphi_1 = \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$$

$$\varphi_2 = (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$$

$$\varphi_3 = \neg\neg\neg p \rightarrow \neg p$$

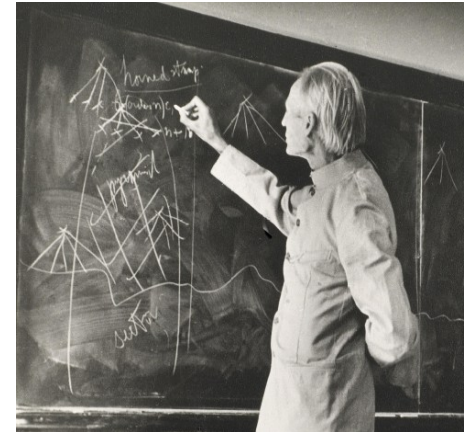
gültig sind.

# Intuitionistic logic

- *Classical logic* is about **truth**
- *Intuitionistic logic* is about **constructibility**
- Intuitionistic logic is therefore also called *constructive logic*
- All statements are classically regarded as either true or false:  $p \vee \neg p$  holds (*tertium non datur*)
  - „there is 7 seven’s somewhere in the decimal representation of the number  $\pi$ “
  - „there are irrational  $p$  and  $q$  such that  $p^q$  is rational“
- Intuitionistic logic, in contrast, requires that we can *construct* the objects whose existence we assert and hence rejects the *tertium non datur* principle

# Intuitionistic logic

- **L.E.J. Brouwer** (1881-1966)
- **Arend Heyting** (1898-1980)
  
- Glivenko
- Kolmogorov
- Gentzen
- Gödel
- ...



# BHK-interpretation (informal constructive semantics)

- *A construction of  $\varphi_1 \wedge \varphi_2$  consists of a construction of  $\varphi_1$  and a construction of  $\varphi_2$ ;*
- *A construction of  $\varphi_1 \vee \varphi_2$  consists of a number  $i \in \{1, 2\}$  and a construction of  $\varphi_i$ ;*
- *A construction of  $\varphi_1 \rightarrow \varphi_2$  is a method (function) transforming every construction of  $\varphi_1$  into a construction of  $\varphi_2$ ;*
- *There is no possible construction of  $\perp$  (where  $\perp$  denotes falsity).*

# BHK-interpretation (informal constructive semantics)

- *A construction of  $\varphi_1 \wedge \varphi_2$  consists of a construction of  $\varphi_1$  and a construction of  $\varphi_2$ ;*
- *A construction of  $\varphi_1 \vee \varphi_2$  consists of a number  $i \in \{1, 2\}$  and a construction of  $\varphi_i$ ;*
- *A construction of  $\varphi_1 \rightarrow \varphi_2$  is a method (function) transforming every construction of  $\varphi_1$  into a construction of  $\varphi_2$ ;*
- *There is no possible construction of  $\perp$  (where  $\perp$  denotes falsity).*

Negation  $\neg\varphi$  is best understood as an abbreviation of an implication  $\varphi \rightarrow \perp$ . That is, we assert  $\neg\varphi$  when the assumption of  $\varphi$  leads to an absurd. It follows that

- *A construction of  $\neg\varphi$  is a method that turns every construction of  $\varphi$  into a non-existent object.*

# BHK-interpretation (informal constructive semantics)

- *A construction of  $\varphi_1 \wedge \varphi_2$  consists of a construction of  $\varphi_1$  and a construction of  $\varphi_2$ ;*
- *A construction of  $\varphi_1 \vee \varphi_2$  consists of a number  $i \in \{1, 2\}$  and a construction of  $\varphi_i$ ;*
- *A construction of  $\varphi_1 \rightarrow \varphi_2$  is a method (function) transforming every construction of  $\varphi_1$  into a construction of  $\varphi_2$ ;*
- *There is no possible construction of  $\perp$  (where  $\perp$  denotes falsity).*

**Note:** This informal interpretation can be formalized in

- **recursion theory:** Kleene's notion of **realizability**
- **lambda calculus:** the **Curry-Howard isomorphism**



# Natural deduction

$\Phi ::= \perp \mid PV \mid (\Phi \rightarrow \Phi) \mid (\Phi \vee \Phi) \mid (\Phi \wedge \Phi).$

$\Gamma, \varphi \vdash \varphi$  (Ax)

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} (\wedge I)$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} (\wedge E) \quad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} (\vee I) \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi \vdash \rho \quad \Gamma, \psi \vdash \rho \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \rho} (\vee E)$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} (\rightarrow I)$$

$$\frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} (\rightarrow E)$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi} (\perp E)$$

# Proof trees

2.2.5. EXAMPLE. Let  $\Gamma$  abbreviate  $\{\varphi \rightarrow (\psi \rightarrow \vartheta), \varphi \rightarrow \psi, \varphi\}$ .

(i)

$$\frac{\varphi \vdash \varphi}{\vdash \varphi \rightarrow \varphi} (\rightarrow \text{I})$$

(ii)

$$\frac{\frac{\varphi, \psi \vdash \varphi}{\varphi \vdash \psi \rightarrow \varphi} (\rightarrow \text{I})}{\vdash \varphi \rightarrow (\psi \rightarrow \varphi)} (\rightarrow \text{I})$$

(iii)

$$\frac{(\rightarrow \text{E}) \frac{\Gamma \vdash \varphi \rightarrow (\psi \rightarrow \vartheta) \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi \rightarrow \vartheta} \quad \frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} (\rightarrow \text{E})}{\Gamma \vdash \vartheta} (\rightarrow \text{E})$$

$$\frac{\frac{\frac{\frac{\varphi \rightarrow (\psi \rightarrow \vartheta), \varphi \rightarrow \psi \vdash \varphi \rightarrow \vartheta}{\varphi \rightarrow (\psi \rightarrow \vartheta) \vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \vartheta)} (\rightarrow \text{I})}{\vdash (\varphi \rightarrow (\psi \rightarrow \vartheta)) \rightarrow (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \vartheta)} (\rightarrow \text{I})}{\Gamma \vdash \vartheta} (\rightarrow \text{E})$$

# Examples of classical tautologies

1.  $\perp \rightarrow p$ ;

2.  $((p \rightarrow q) \rightarrow p) \rightarrow p$ ;

3.  $p \rightarrow \neg\neg p$ ;

4.  $\neg\neg p \rightarrow p$ ;

5.  $\neg\neg\neg p \rightarrow \neg p$ ;

6.  $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$ ;

7.  $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ ;

8.  $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ ;

9.  $(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$ ;

10.  $((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow (p \leftrightarrow (q \leftrightarrow r))$ ;

11.  $((p \wedge q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$ ;

12.  $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ ;

13.  $\neg\neg(p \vee \neg p)$ .

Not all of these are intuitionistic theorems!

# Example

$$\boxed{\neg(p \vee q) \rightarrow \neg p \wedge \neg q}$$

Write  $\neg p$  as  $p \rightarrow 0$ . We are to prove  $((p \vee q) \rightarrow 0) \rightarrow ((p \rightarrow 0) \wedge (q \rightarrow 0))$

$$\begin{array}{l} (p \vee q) \rightarrow 0, p \vdash p \quad (\text{Ax}) \\ \hline (p \vee q) \rightarrow 0, p \vdash p \vee q \quad (\vee\text{I}) \\ (p \vee q) \rightarrow 0, p \vdash p \vee q \quad (p \vee q) \rightarrow 0, p \vdash (p \vee q) \rightarrow 0 \quad (\text{Ax}) \\ \hline (p \vee q) \rightarrow 0, p \vdash 0 \quad (\rightarrow\text{E}) \\ \hline (p \vee q) \rightarrow 0 \vdash p \rightarrow 0 \quad (\rightarrow\text{I}) \end{array}$$

Similarly, we can prove  $(p \vee q) \rightarrow 0 \vdash q \rightarrow 0$  by starting from  $(p \vee q) \rightarrow 0, q \vdash q$ .

Therefore we have, by  $(\wedge\text{I})$   $(p \vee q) \rightarrow 0 \vdash (p \rightarrow 0) \wedge (q \rightarrow 0)$ .

Then, by  $(\rightarrow\text{I})$   $\vdash ((p \vee q) \rightarrow 0) \rightarrow ((p \rightarrow 0) \wedge (q \rightarrow 0))$ .

# Example

$$\boxed{(-p \wedge -q) \rightarrow -(p \vee q)}$$

Write  $\neg p$  as  $p \rightarrow 0$ . We are to prove  $((p \rightarrow 0) \wedge (q \rightarrow 0)) \rightarrow ((p \vee q) \rightarrow 0)$

$$\frac{((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q), p \vdash p \text{ (Ax)} \quad ((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q), p \vdash p \rightarrow 0 \text{ (\wedge E)}}{-----} \quad (\rightarrow E)$$

[1]  $((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q), p \vdash 0$

Similarly, [2]  $((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q), q \vdash 0$ .

Now, we have [3]  $((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q) \vdash p \vee q \text{ (Ax)}$ .

By [1], [2], [3] we conclude  $((p \rightarrow 0) \wedge (q \rightarrow 0), (p \vee q) \vdash 0$  by (vE).

Therefore, by ( $\rightarrow$ E) conclude  $((p \rightarrow 0) \wedge (q \rightarrow 0) \vdash (p \vee q) \rightarrow 0$ .

Finally, by ( $\rightarrow$ E)  $\vdash ((p \rightarrow 0) \wedge (q \rightarrow 0)) \rightarrow ((p \vee q) \rightarrow 0)$ .

# Example

$$\boxed{\text{--- } p \rightarrow \neg p}$$

Write  $\neg p$  as  $p \rightarrow 0$ . We are to prove  $((p \rightarrow 0) \rightarrow 0) \rightarrow (p \rightarrow 0)$

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p, p \rightarrow 0 \vdash p \rightarrow 0$$

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p, p \rightarrow 0 \vdash p$$

---

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p, p \rightarrow 0 \vdash 0$$

---

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p \vdash (p \rightarrow 0) \rightarrow 0$$

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p \vdash ((p \rightarrow 0) \rightarrow 0) \rightarrow 0$$

---

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0, p \vdash 0$$

---

$$((p \rightarrow 0) \rightarrow 0) \rightarrow 0 \vdash p \rightarrow 0$$

---

$$\vdash (((p \rightarrow 0) \rightarrow 0) \rightarrow 0) \rightarrow (p \rightarrow 0)$$