Logische Grundlagen des Software Engineerings

Prof. Dr. Jakob Rehof

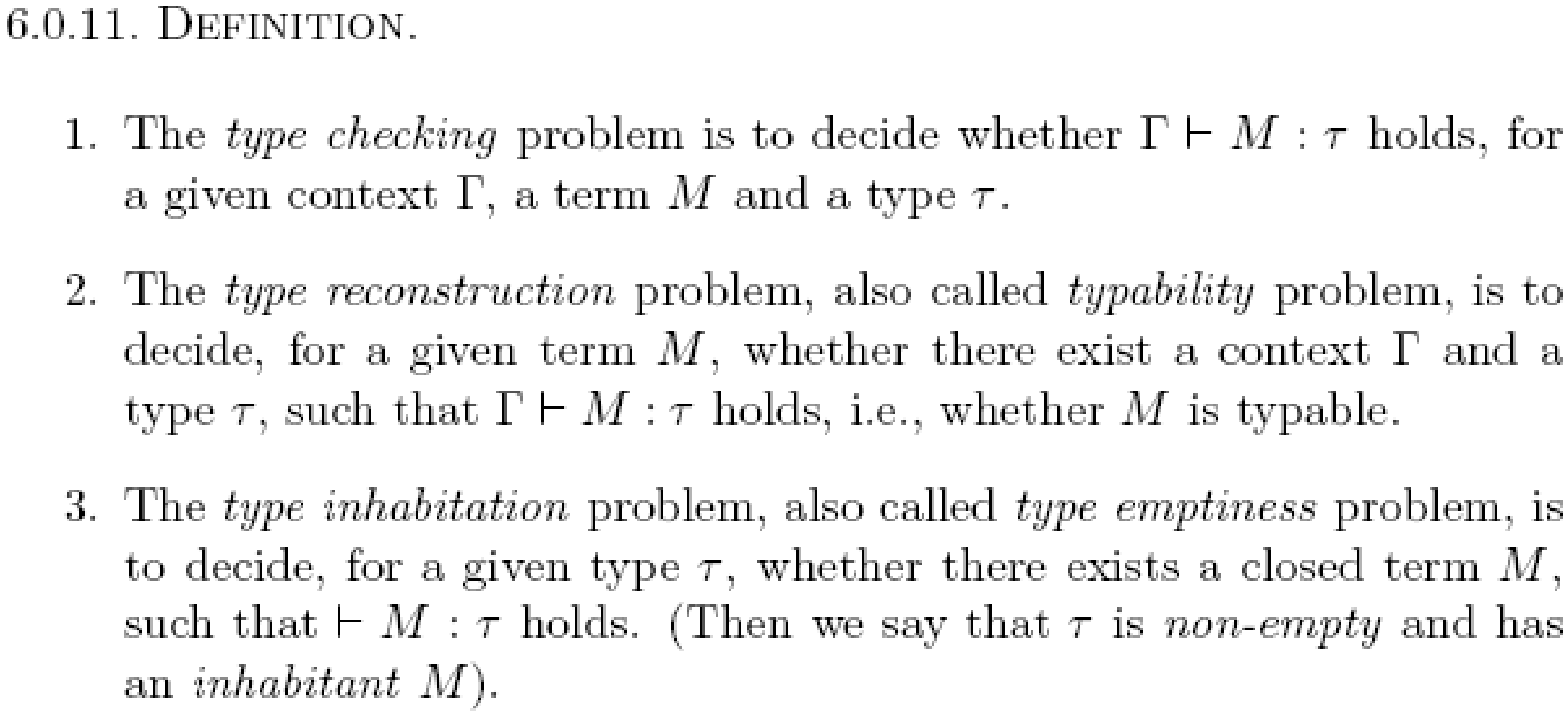
Lehrstuhl 14, Software Engineering

Type checking and related problems

- Decision problems arising from the (mostly Curry-style) ternary predicate

$$\Gamma \mid\!\!- \ M : \tau$$

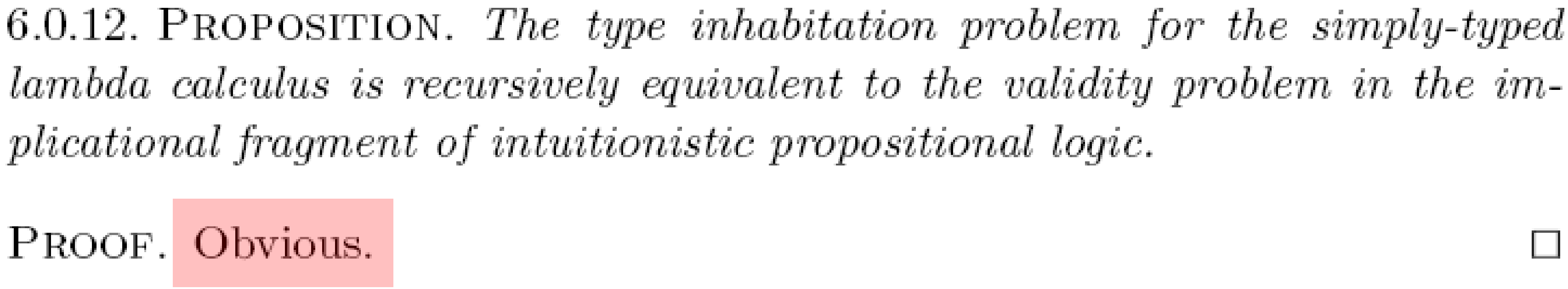
Type checking, reconstruction, inhabitation



6.0.11. DEFINITION.

1. The *type checking* problem is to decide whether $\Gamma \vdash M : \tau$ holds, for a given context Γ, a term M and a type τ.
2. The *type reconstruction* problem, also called *typability* problem, is to decide, for a given term M, whether there exist a context Γ and a type τ, such that $\Gamma \vdash M : \tau$ holds, i.e., whether M is typable.
3. The *type inhabitation* problem, also called *type emptiness* problem, is to decide, for a given type τ, whether there exists a closed term M, such that $\vdash M : \tau$ holds. (Then we say that τ is *non-empty* and has an *inhabitant* M).

Inhabitation and validity



6.0.12. PROPOSITION. *The type inhabitation problem for the simply-typed lambda calculus is recursively equivalent to the validity problem in the implicational fragment of intuitionistic propositional logic.*

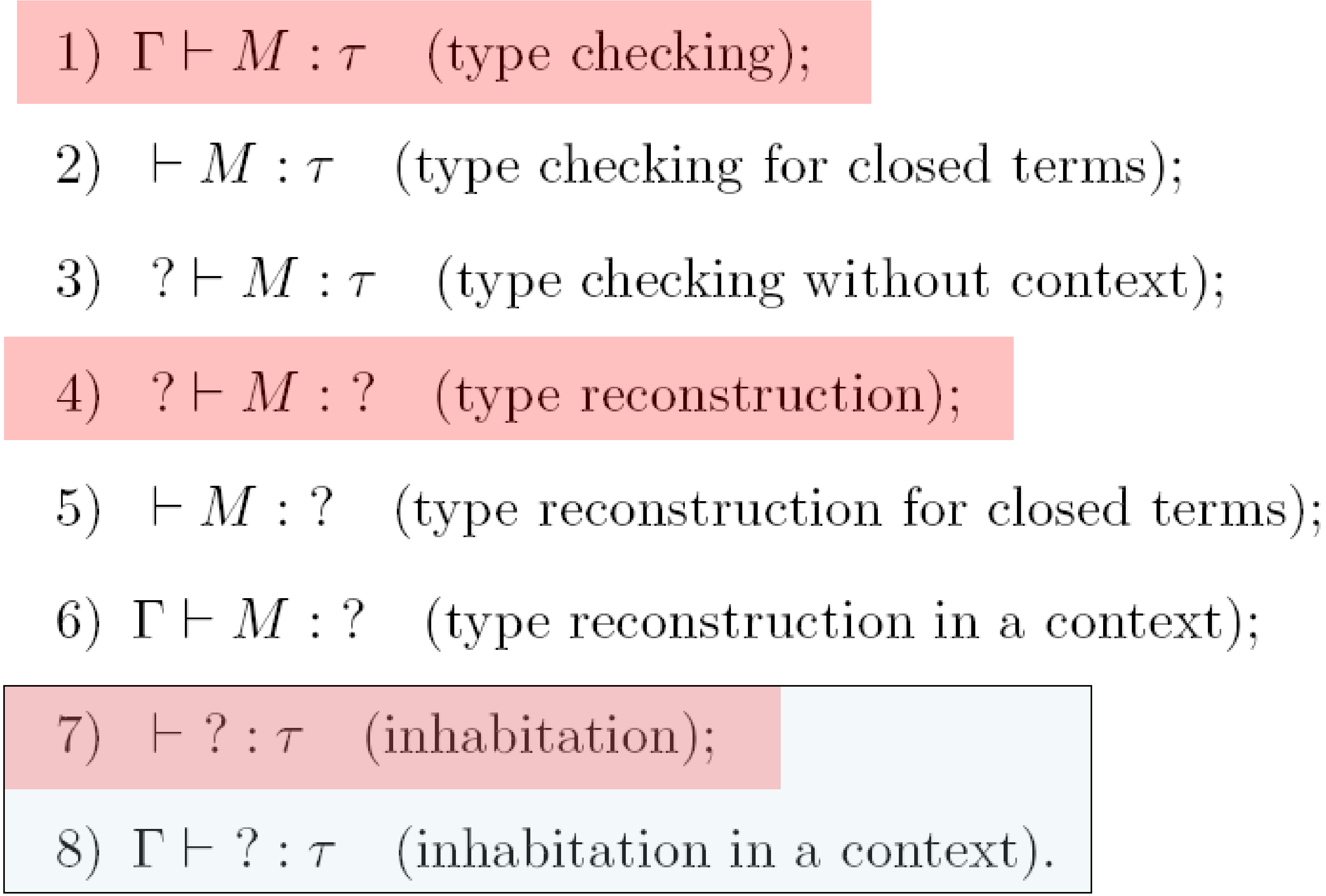
PROOF. Obvious. ☐

Why??

12 variants, of which 4 are trivial ...

- $? \vdash ? : ?;$
- $\Gamma \vdash ? : ?;$
- $\vdash ? : ?;$
- $? \vdash ? : \tau.$

... and 8 are interesting:



1) $\Gamma \vdash M : \tau$ (type checking);

2) $\vdash M : \tau$ (type checking for closed terms);

3) $? \vdash M : \tau$ (type checking without context);

4) $? \vdash M : ?$ (type reconstruction);

5) $\vdash M : ?$ (type reconstruction for closed terms);

6) $\Gamma \vdash M : ?$ (type reconstruction in a context);

7) $\vdash ? : \tau$ (inhabitation);

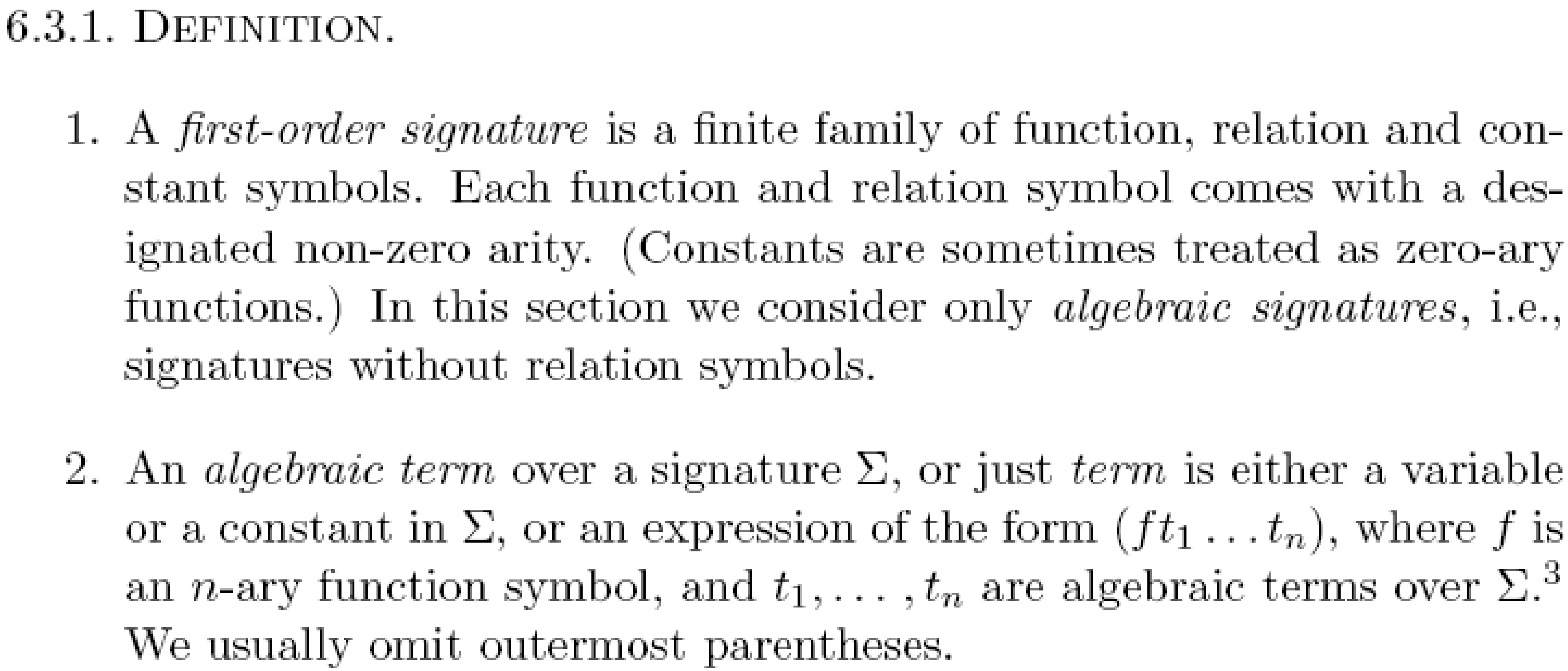
8) $\Gamma \vdash ? : \tau$ (inhabitation in a context).

Unification

- Solving systems of term equations

$$\{ t_i = t'_i \}$$

Terms

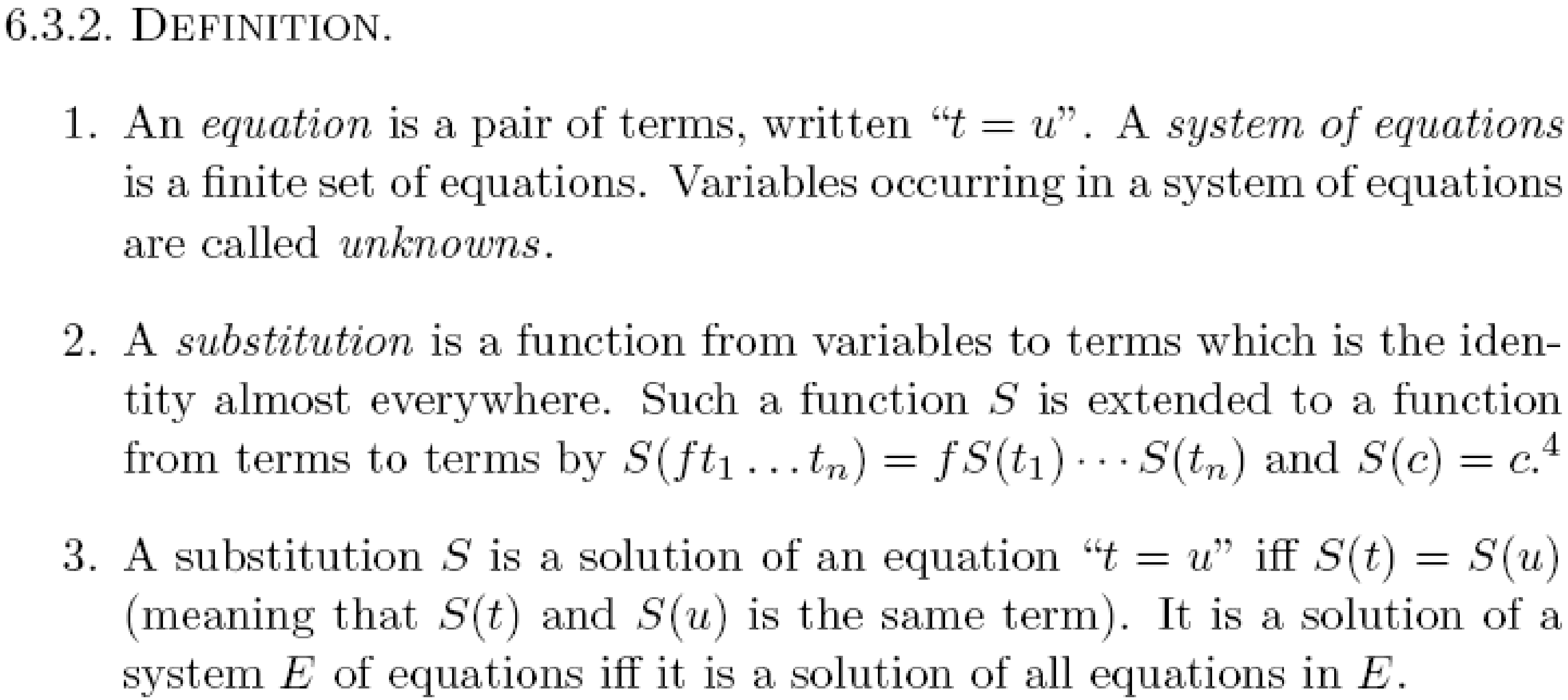


6.3.1. DEFINITION.

1. A *first-order signature* is a finite family of function, relation and constant symbols. Each function and relation symbol comes with a designated non-zero arity. (Constants are sometimes treated as zero-ary functions.) In this section we consider only *algebraic signatures*, i.e., signatures without relation symbols.

2. An *algebraic term* over a signature Σ, or just *term* is either a variable or a constant in Σ, or an expression of the form $(ft_1 \ldots t_n)$, where f is an n-ary function symbol, and $t_1, \ldots, t_n$ are algebraic terms over Σ.[3] We usually omit outermost parentheses.

Equations



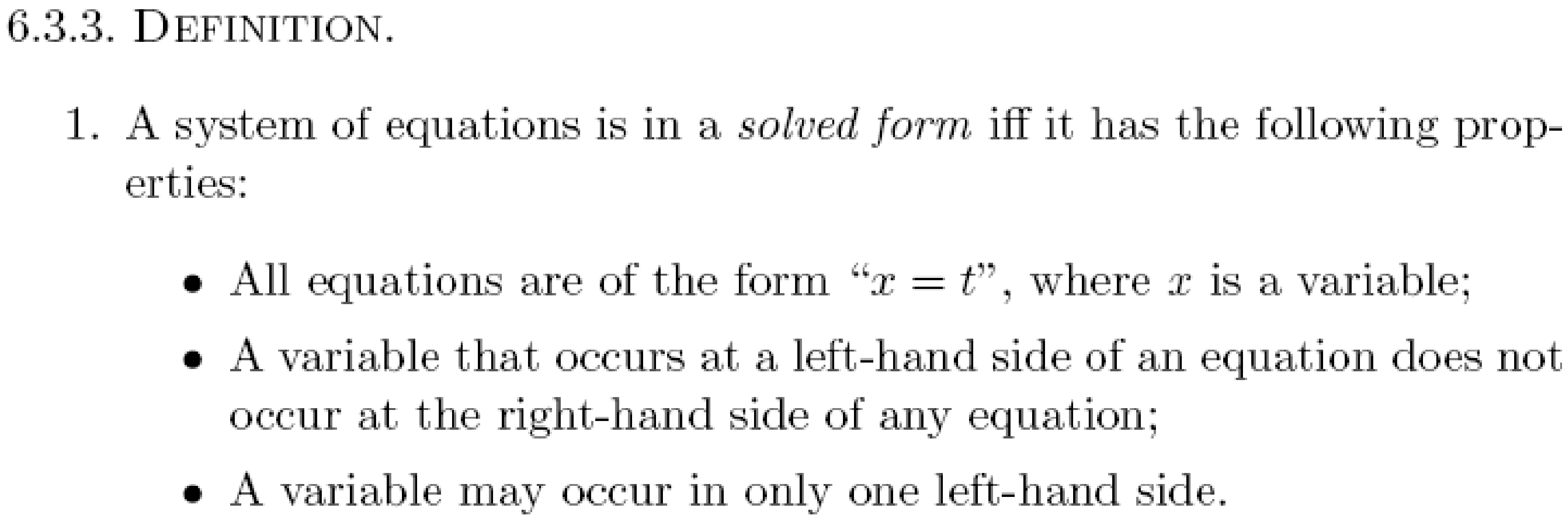
6.3.2. DEFINITION.

1. An *equation* is a pair of terms, written "$t = u$". A *system of equations* is a finite set of equations. Variables occurring in a system of equations are called *unknowns*.

2. A *substitution* is a function from variables to terms which is the identity almost everywhere. Such a function S is extended to a function from terms to terms by $S(ft_1 \ldots t_n) = fS(t_1) \cdots S(t_n)$ and $S(c) = c$.[4]

3. A substitution S is a solution of an equation "$t = u$" iff $S(t) = S(u)$ (meaning that $S(t)$ and $S(u)$ is the same term). It is a solution of a system E of equations iff it is a solution of all equations in E.

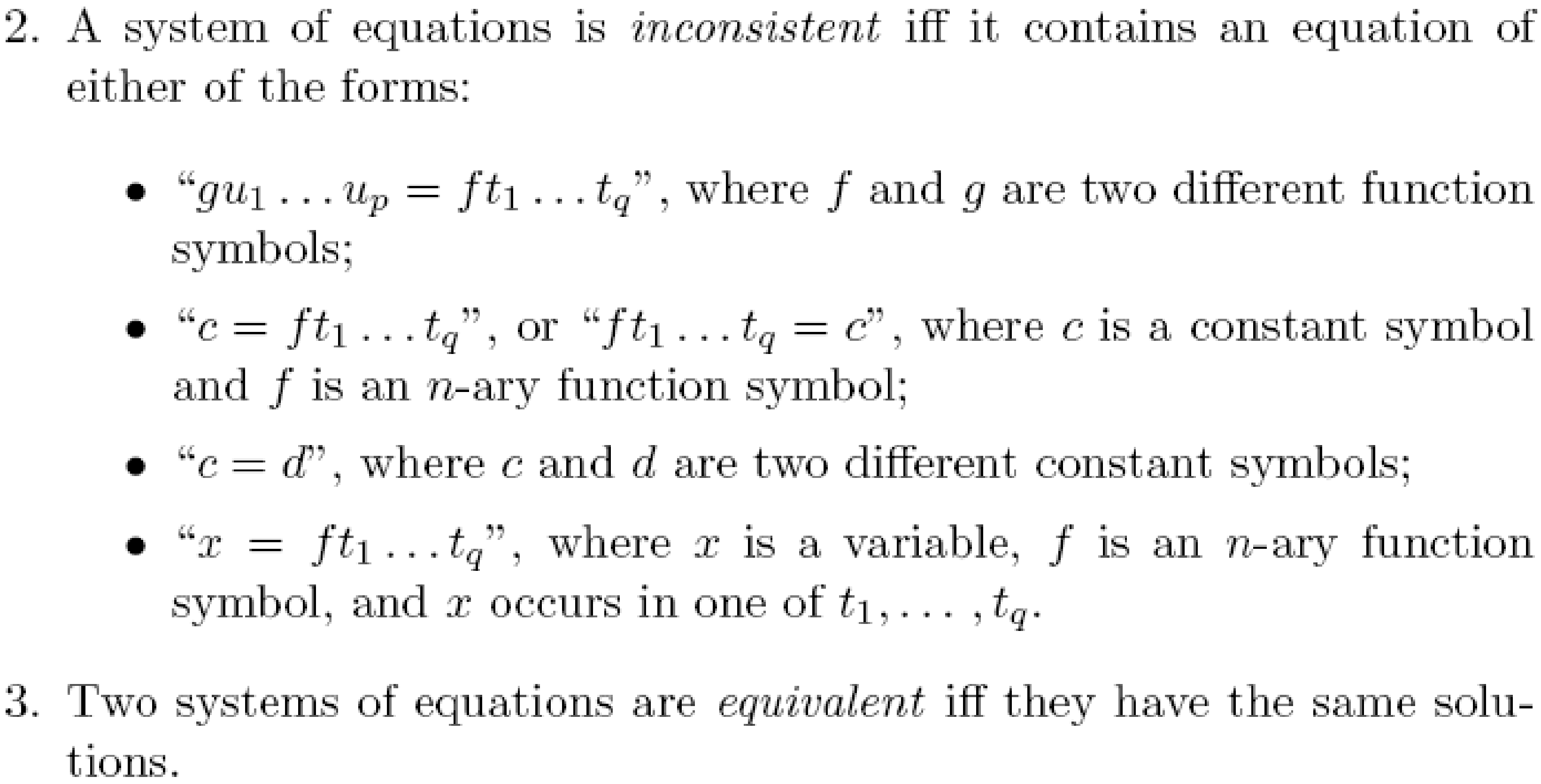
Equations



6.3.3. DEFINITION.

1. A system of equations is in a *solved form* iff it has the following properties:
   - All equations are of the form "$x = t$", where x is a variable;
   - A variable that occurs at a left-hand side of an equation does not occur at the right-hand side of any equation;
   - A variable may occur in only one left-hand side.

Equations



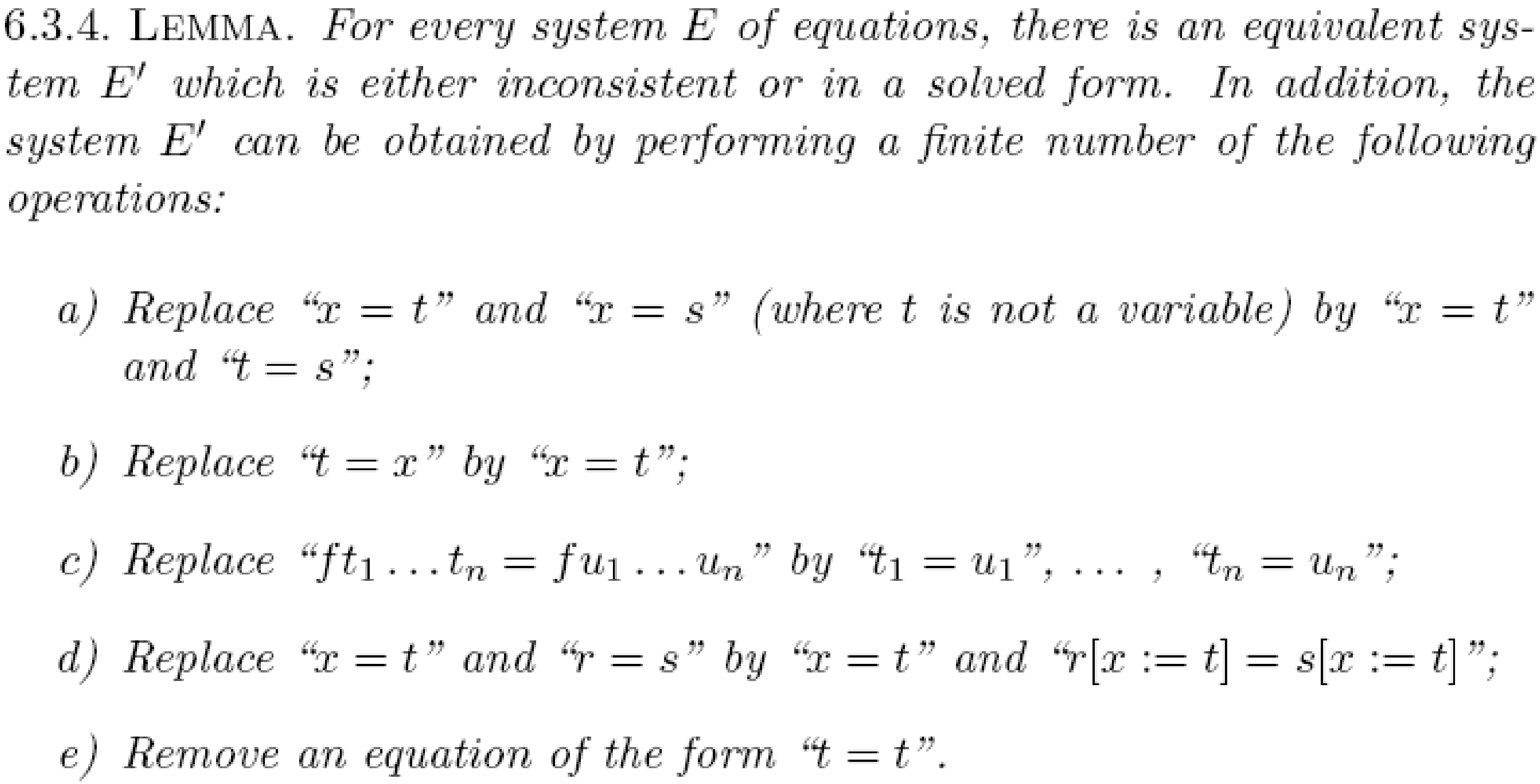
2. A system of equations is *inconsistent* iff it contains an equation of either of the forms:
   - "$gu_1 \ldots u_p = ft_1 \ldots t_q$", where f and g are two different function symbols;
   - "$c = ft_1 \ldots t_q$", or "$ft_1 \ldots t_q = c$", where c is a constant symbol and f is an n-ary function symbol;
   - "$c = d$", where c and d are two different constant symbols;
   - "$x = ft_1 \ldots t_q$", where x is a variable, f is an n-ary function symbol, and x occurs in one of $t_1, \ldots, t_q$.
3. Two systems of equations are *equivalent* iff they have the same solutions.

Equations

It is easy to see that an inconsistent system has no solutions and that a solved system E has a solution S_0 defined as follows:

- If a variable x is undefined then $S_0(x) = x$;
- If "$x = t$" is in E, then $S_0(x) = t$.

Unification algorithm (Robinson)



6.3.4. Lemma. *For every system E of equations, there is an equivalent system E' which is either inconsistent or in a solved form. In addition, the system E' can be obtained by performing a finite number of the following operations:*

a) Replace "$x = t$" and "$x = s$" (where t is not a variable) by "$x = t$" and "$t = s$";

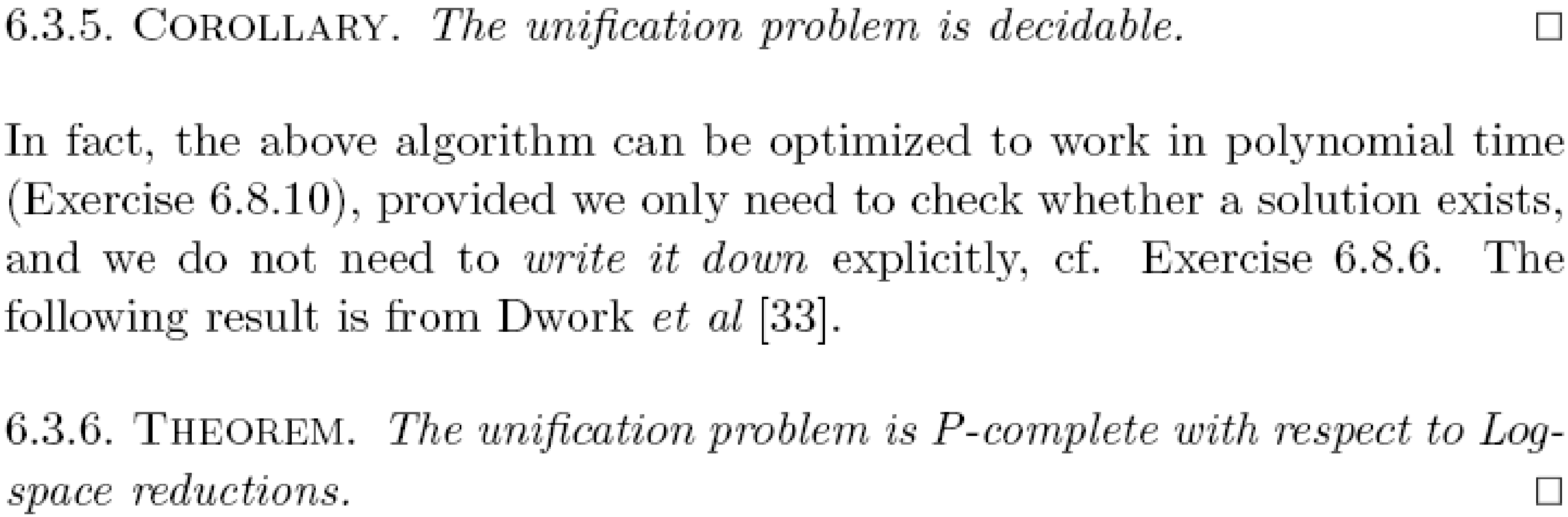
b) Replace "$t = x$" by "$x = t$";

c) Replace "$ft_1 \dots t_n = fu_1 \dots u_n$" by "$t_1 = u_1$", … , "$t_n = u_n$";

d) Replace "$x = t$" and "$r = s$" by "$x = t$" and "$r[x := t] = s[x := t]$";

e) Remove an equation of the form "$t = t$".

Unification algorithm (Robinson)

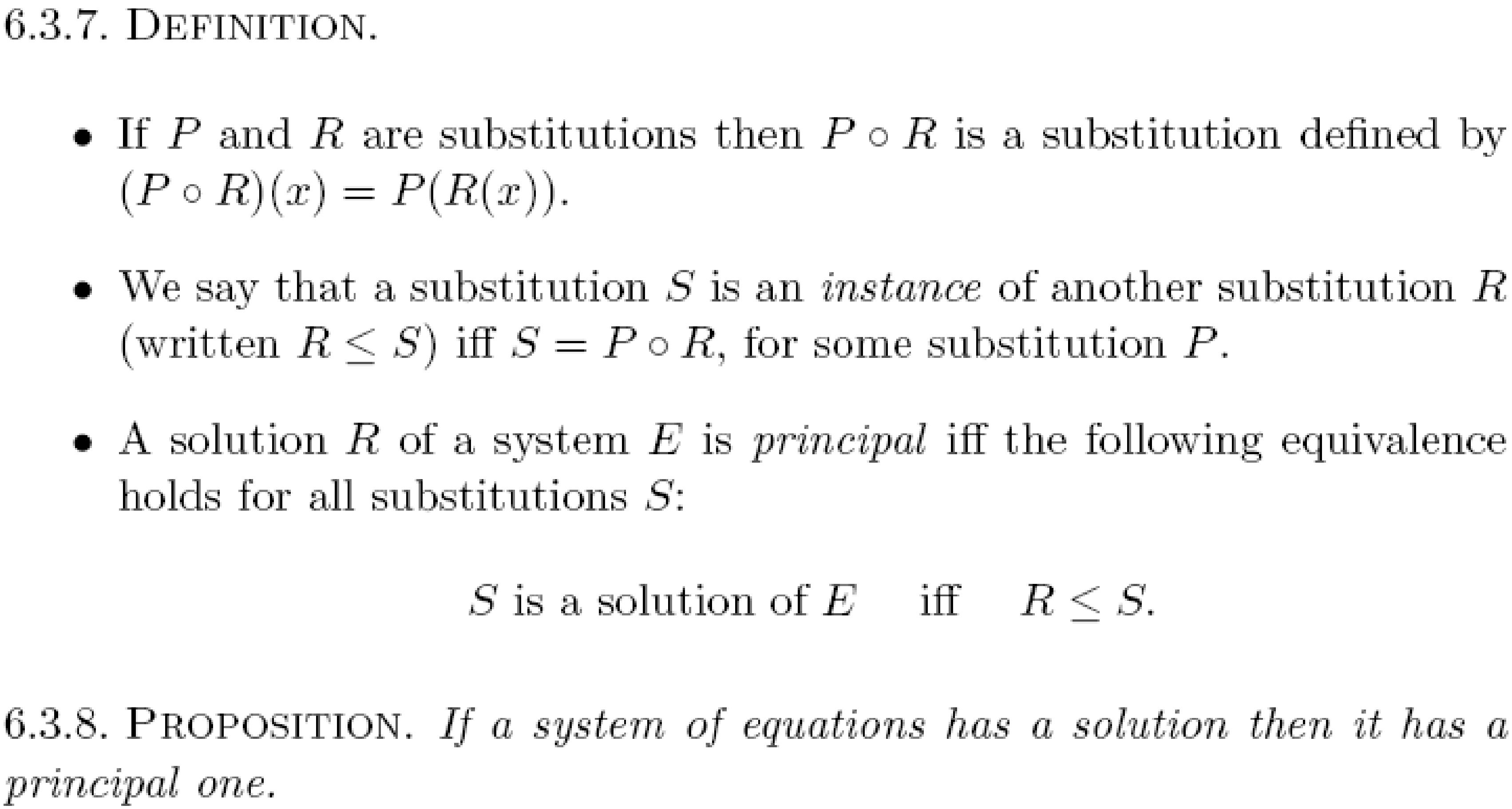


6.3.5. Corollary. *The unification problem is decidable.* □

In fact, the above algorithm can be optimized to work in polynomial time (Exercise 6.8.10), provided we only need to check whether a solution exists, and we do not need to *write it down* explicitly, cf. Exercise 6.8.6. The following result is from Dwork *et al* [33].

6.3.6. Theorem. *The unification problem is P-complete with respect to Log-space reductions.* □

Principal (most general) solution



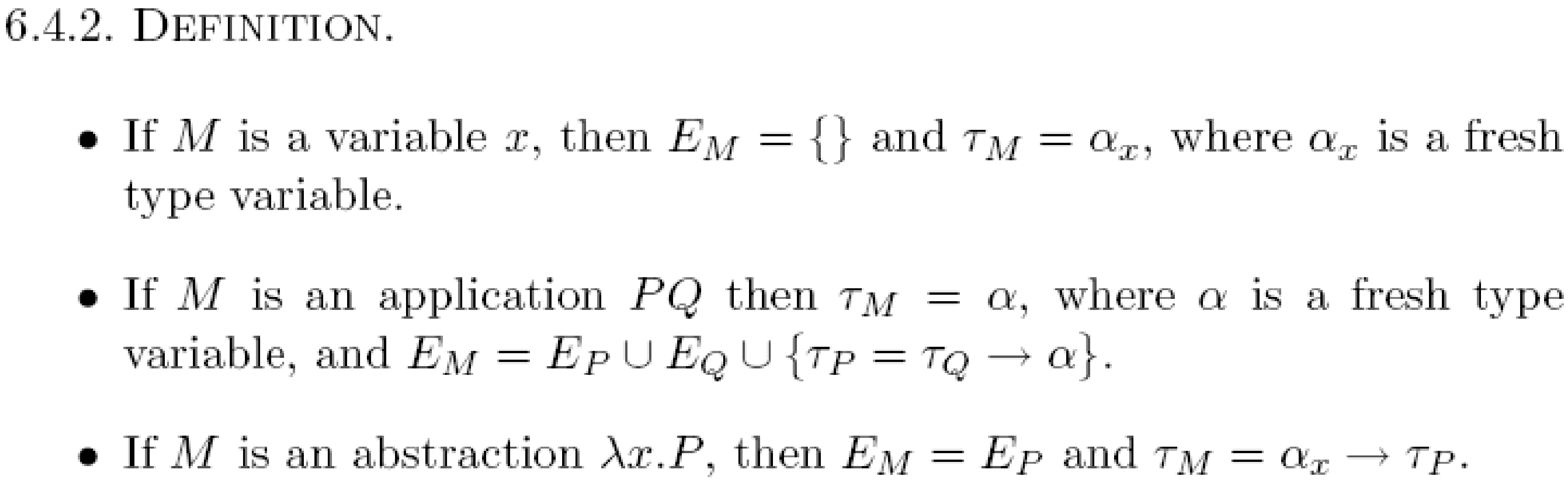
6.3.7. DEFINITION.

- If P and R are substitutions then $P \circ R$ is a substitution defined by $(P \circ R)(x) = P(R(x))$.
- We say that a substitution S is an *instance* of another substitution R (written $R \leq S$) iff $S = P \circ R$, for some substitution P.
- A solution R of a system E is *principal* iff the following equivalence holds for all substitutions S:

$$S \text{ is a solution of } E \quad \text{iff} \quad R \leq S.$$

6.3.8. PROPOSITION. *If a system of equations has a solution then it has a principal one.*

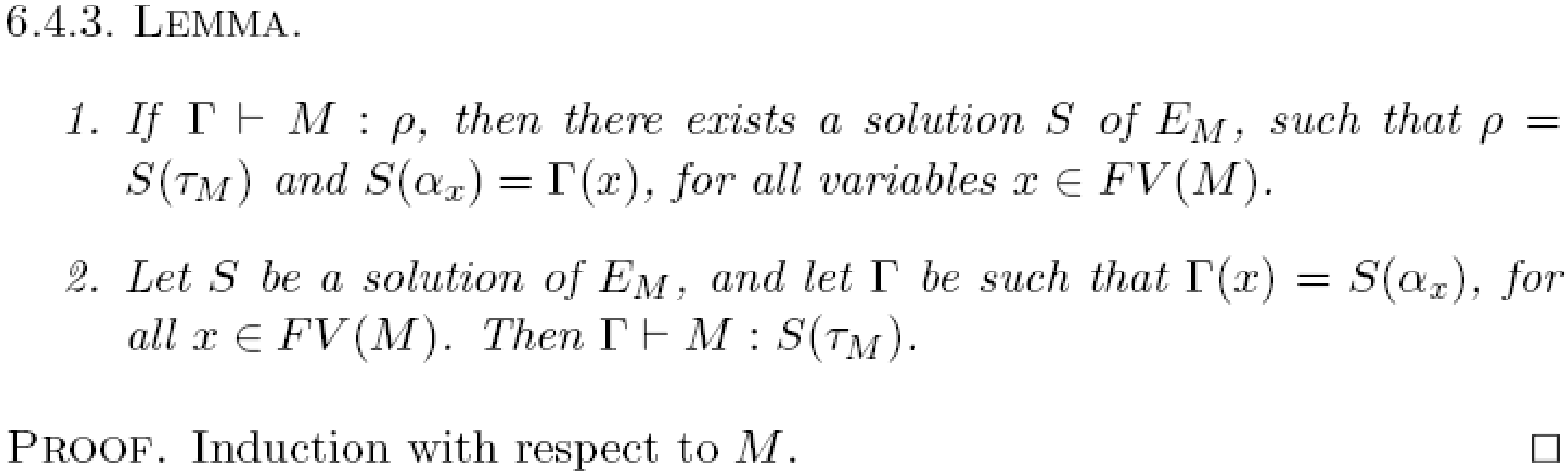
Type reconstruction



6.4.2. Definition.

- If M is a variable x, then $E_M = \{\}$ and $\tau_M = \alpha_x$, where α_x is a fresh type variable.
- If M is an application PQ then $\tau_M = \alpha$, where α is a fresh type variable, and $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \to \alpha\}$.
- If M is an abstraction $\lambda x.P$, then $E_M = E_P$ and $\tau_M = \alpha_x \to \tau_P$.

Type reconstruction

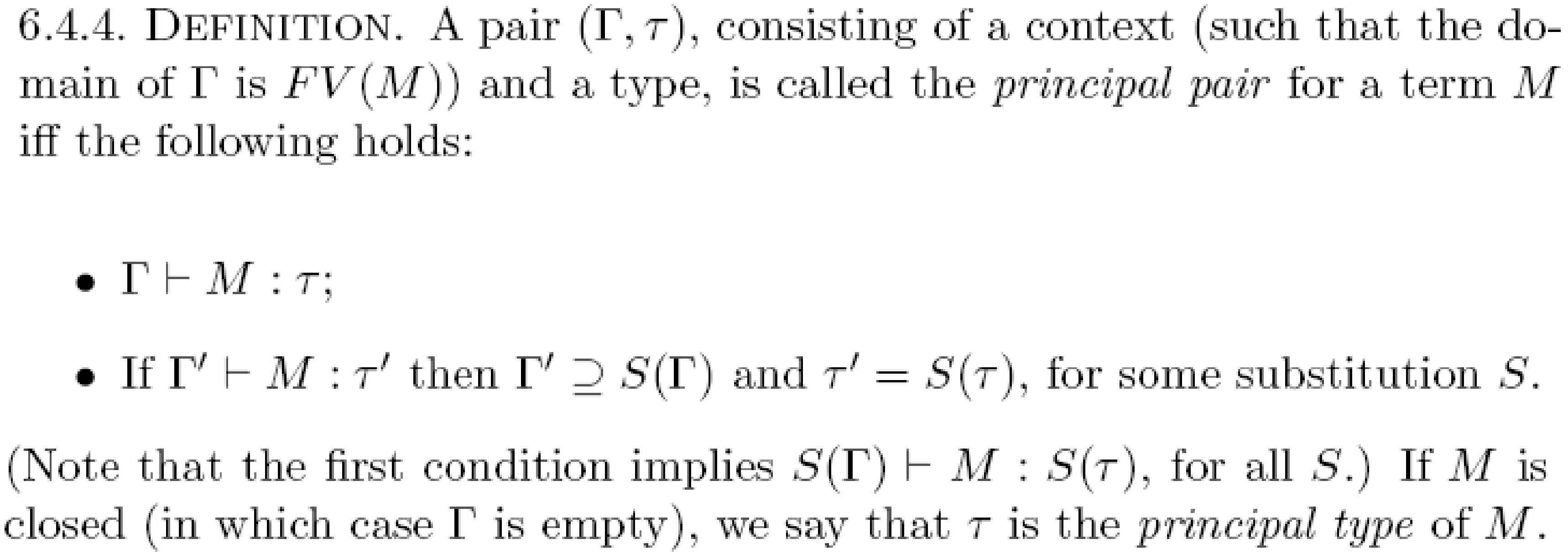


6.4.3. LEMMA.

1. *If* $\Gamma \vdash M : \rho$, *then there exists a solution* S *of* E_M, *such that* $\rho = S(\tau_M)$ *and* $S(\alpha_x) = \Gamma(x)$, *for all variables* $x \in FV(M)$.
2. *Let* S *be a solution of* E_M, *and let* Γ *be such that* $\Gamma(x) = S(\alpha_x)$, *for all* $x \in FV(M)$. *Then* $\Gamma \vdash M : S(\tau_M)$.

PROOF. Induction with respect to M. $\square$

Principal pair, principal type

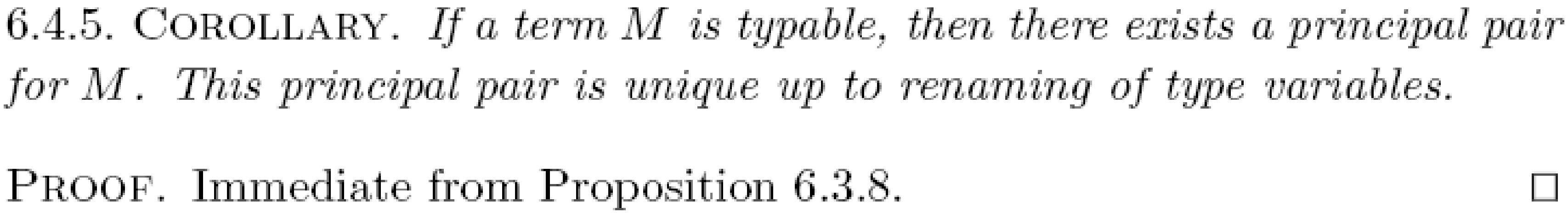


6.4.4. Definition. A pair (Γ, τ), consisting of a context (such that the domain of Γ is $FV(M)$) and a type, is called the *principal pair* for a term M iff the following holds:

- $\Gamma \vdash M : \tau$;
- If $\Gamma' \vdash M : \tau'$ then $\Gamma' \supseteq S(\Gamma)$ and $\tau' = S(\tau)$, for some substitution S.

(Note that the first condition implies $S(\Gamma) \vdash M : S(\tau)$, for all S.) If M is closed (in which case Γ is empty), we say that τ is the *principal type* of M.

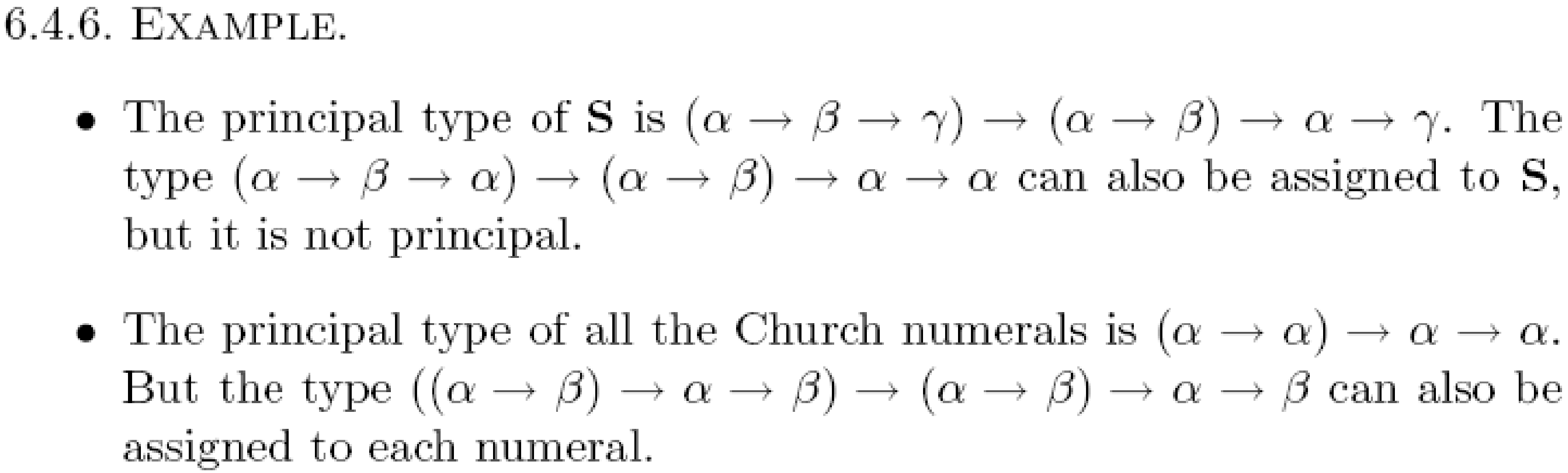
Principal type theorem



6.4.5. Corollary. *If a term M is typable, then there exists a principal pair for M. This principal pair is unique up to renaming of type variables.*

Proof. Immediate from Proposition 6.3.8. □

Example



6.4.6. EXAMPLE.

- The principal type of **S** is $(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$. The type $(\alpha \to \beta \to \alpha) \to (\alpha \to \beta) \to \alpha \to \alpha$ can also be assigned to **S**, but it is not principal.
- The principal type of all the Church numerals is $(\alpha \to \alpha) \to \alpha \to \alpha$. But the type $((\alpha \to \beta) \to \alpha \to \beta) \to (\alpha \to \beta) \to \alpha \to \beta$ can also be assigned to each numeral.