LMSE
Logische Methoden des Software Engineerings

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Diese Vorlesung

- Intuitionistische Logik
- Natürliche Deduktion

Lesen und Übungen

- Lesen Kapitel 2.1 und 2.2 (Seite 23 - 28)
- Übungen
  - Aufgabe 2.7.3
  - Zeigen Sie mittels natürlicher Deduktion, dass die Formeln

$$\varphi_1 = \neg(p \lor q) \to (\neg p \land \neg q)$$

$$\varphi_2 = (\neg p \land \neg q) \to \neg(p \lor q)$$

$$\varphi_3 = \neg\neg\neg p \to \neg p$$

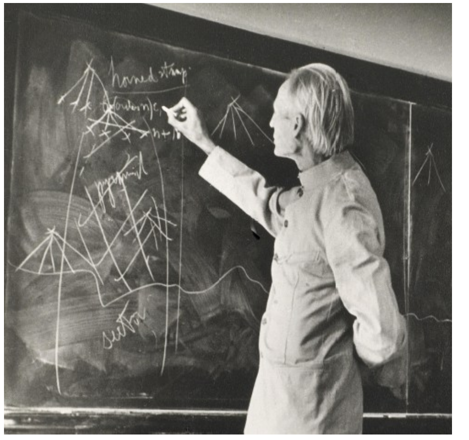
gültig sind.

Intuitionistic logic

- *Classical logic* is about **truth**
- *Intuitionistic logic* is about **constructibility**
- Intuitionistic logic is therefore also called *constructive logic*
- All statements are classically regarded as either true or false: **p v –p** holds (*tertium non datur*)
  – „there is 7 seven‘s somewhere in the decimal representation of the number π“
  – „there are irrational p and q such that p^q is rational“

- Intuitionistic logic, in contrast, requires that we can *construct* the objects whose existence we assert and hence rejects the *tertium non datur* principle

Intuitionistic logic



- **L.E.J.Brouwer** (1881-1966)
- **Arend Heyting** (1898-1980)

- Glivenko
- Kolmogorov
- Gentzen
- Gödel
- ...

BHK-interpretation (informal constructive semantics)

- *A construction of* $\varphi_1 \wedge \varphi_2$ *consists of a construction of* φ_1 *and a construction of* φ_2*;*
- *A construction of* $\varphi_1 \vee \varphi_2$ *consists of a number* $i \in \{1,2\}$ *and a construction of of* φ_i*;*
- *A construction of* $\varphi_1 \to \varphi_2$ *is a method (function) transforming every construction of* φ_1 *into a construction of* φ_2*;*
- *There is no possible construction of* $\bot$ *(where* $\bot$ *denotes falsity).*

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Negation $\neg\varphi$ is best understood as an abbreviation of an implication $\varphi \to \bot$. That is, we assert $\neg\varphi$ when the assumption of φ leads to an absurd. It follows that

- *A construction of* $\neg\varphi$ *is a method that turns every construction of* φ *into a non-existent object.*

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- *A construction of* $\varphi_1 \to \varphi_2$ *is a method (function) transforming every construction of* φ_1 *into a construction of* φ_2*;*
- *There is no possible construction of* $\perp$ *(where* $\perp$ *denotes falsity).*

Note: This informal interpretation can be formalized in

- **recursion theory:** Kleene‘s notion of **realizability**
- **lambda calculus:** the **Curry-Howard isomorphism**

Natural deduction

$$\Phi ::= \bot \mid PV \mid (\Phi \to \Phi) \mid (\Phi \vee \Phi) \mid (\Phi \wedge \Phi).$$

$$\Gamma, \varphi \vdash \varphi \ (\mathrm{Ax})$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \ (\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} (\wedge\mathrm{E}) \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \ (\vee\mathrm{I}) \ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \rho \quad \Gamma, \psi \vdash \rho \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \rho} \ (\vee\mathrm{E})$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to \mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} (\to \mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \ (\bot\mathrm{E})$$

Proof trees

2.2.5. EXAMPLE. Let Γ abbreviate $\{\varphi \to (\psi \to \vartheta), \varphi \to \psi, \varphi\}$.

(i)

$$\dfrac{\varphi \vdash \varphi}{\vdash \varphi \to \varphi}\;(\to \mathrm{I})$$

(ii)

$$\dfrac{\dfrac{\varphi, \psi \vdash \varphi}{\varphi \vdash \psi \to \varphi}\;(\to \mathrm{I})}{\vdash \varphi \to (\psi \to \varphi)}\;(\to \mathrm{I})$$

(iii)

$$\dfrac{\dfrac{\dfrac{\dfrac{(\to \mathrm{E})\dfrac{\Gamma \vdash \varphi \to (\psi \to \vartheta) \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi \to \vartheta} \qquad \dfrac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}(\to \mathrm{E})}{\Gamma \vdash \vartheta}(\to \mathrm{E})}{\varphi \to (\psi \to \vartheta), \varphi \to \psi \vdash \varphi \to \vartheta}(\to \mathrm{I})}{\varphi \to (\psi \to \vartheta) \vdash (\varphi \to \psi) \to (\varphi \to \vartheta)}(\to \mathrm{I})}{\vdash (\varphi \to (\psi \to \vartheta)) \to (\varphi \to \psi) \to (\varphi \to \vartheta)}(\to \mathrm{I})$$

Examples of classical tautologies

1. $\bot \to p$;
2. $((p \to q) \to p) \to p$;
3. $p \to \neg\neg p$;
4. $\neg\neg p \to p$;
5. $\neg\neg\neg p \to \neg p$;
6. $(\neg q \to \neg p) \to (p \to q)$;
7. $(p \to q) \to (\neg q \to \neg p)$;
8. $\neg(p \land q) \to (\neg p \lor \neg q)$;
9. $(\neg p \lor \neg q) \to \neg(p \land q)$;
10. $((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow (p \leftrightarrow (q \leftrightarrow r))$;
11. $((p \land q) \to r) \leftrightarrow (p \to (q \to r))$;
12. $(p \to q) \leftrightarrow (\neg p \lor q)$;
13. $\neg\neg(p \lor \neg p)$.

Not all of these are intuitionistic theorems!

Example

$-(p \vee q) \rightarrow -p \wedge -q$

Write $–p$ as $p \rightarrow 0$. We are to prove $((p \vee q) \rightarrow 0) \rightarrow ((p \rightarrow 0) \wedge (q \rightarrow 0))$

$(p \vee q) \rightarrow 0, p \vdash p$ (Ax)

---------------------------------- (vI)

$(p \vee q) \rightarrow 0, p \vdash p \vee q$ $\quad\quad$ $(p \vee q) \rightarrow 0, p \vdash (p \vee q) \rightarrow 0$ (Ax)

-- ($\rightarrow$E)

$(p \vee q) \rightarrow 0, p \vdash 0$

-- ($\rightarrow$I)

$(p \vee q) \rightarrow 0 \vdash p \rightarrow 0$

Similarly, we can prove $(p \vee q) \rightarrow 0 \vdash q \rightarrow 0$ by starting from $(p \vee q) \rightarrow 0, q \vdash q$.

Therefore we have, by ($\wedge$I) $(p \vee q) \rightarrow 0 \vdash (p \rightarrow 0) \wedge (q \rightarrow 0)$.

Then, by ($\rightarrow$I) $\vdash ((p \vee q) \rightarrow 0) \rightarrow ((p \rightarrow 0) \wedge (q \rightarrow 0))$.

Example

(-p /\ -q) → -(p v q)

Write –p as p→0. We are to prove ((p →0) /\ (q → 0)) → ((p v q) → 0)

((p →0) /\ (q → 0), (p v q) , p |- p (Ax) ((p →0) /\ (q → 0), (p v q) , p |- p → 0 (/\E)
--- (→E)
[1] ((p →0) /\ (q → 0), (p v q) , p |- 0

Similarly, [2] ((p →0) /\ (q → 0), (p v q) , q |- 0.

Now, we have [3] ((p →0) /\ (q → 0), (p v q) |- p v q (Ax).

By [1], [2], [3] we conclude ((p →0) /\ (q → 0), (p v q) |- 0 by (vE).

Therefore, by (→E) conclude ((p →0) /\ (q → 0) |- (p v q) → 0.

Finally, by (→E) |- ((p →0) /\ (q → 0)) → ((p v q) → 0).

Example

--- p → -p

Write –p as p→0. We are to prove (((p→0)→0)→0) → (p→0)

```
((p→0)→0)→0, p, p→0 |-  p→0                 ((p→0)→0)→0, p, p→0  |- p
--------------------------------------------------------------------------------
                ((p→0)→0)→0, p, p→0  |- 0
        ------------------------------------------------
               ((p→0)→0)→0, p  |-  (p→0)→0      ((p→0)→0)→0, p  |-  ((p→0)→0)→0
        ------------------------------------------------------------------------------------------------
                                ((p→0)→0)→0, p  |- 0
                              -------------------------------------
                                ((p→0)→0)→0  |-  p→0
                              ----------------------------------------
                               |- (((p→0)→0)→0) → (p→0)
```