Intersection Type Calculi of Bounded Dimension

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Intersection types [CDC80; CDCV81; BCDC83] allow terms to be assigned multiple types $\Gamma \vdash M : A_1 \cap \ldots \cap A_n$.
Intersection Type Inhabitation

- *Intersection types* [CDC80; CDCV81; BCDC83] allow terms to be assigned multiple types $\Gamma \vdash M : A_1 \cap \ldots \cap A_n$

- Due to their enormous expressive power, decision problems (typability, inhabitation) are undecidable
Intersection Type Inhabitation

- **Intersection types** [CDC80; CDCV81; BCDC83] allow terms to be assigned multiple types $\Gamma \vdash M : A_1 \cap \ldots \cap A_n$

- Due to their enormous expressive power, decision problems (typability, inhabitation) are undecidable

- We are concerned with understanding the borderline between undecidability and decidability for the *inhabitation problem*:
  - $\Gamma \vdash ? : \sigma$
    - *Given an environment $\Gamma$ and a type $\sigma$, is there a term $M$ such that $\Gamma \vdash M : \sigma$?*
Some Motivations

- $\Gamma \vdash ? : \sigma$
  
  Given an environment $\Gamma$ and a type $\sigma$, is there a term $M$ such that $\Gamma \vdash M : \sigma$?

- Constructively, this is a program *synthesis* problem. Intersection types have been proposed as an important foundation for type theoretic synthesis [Düd+12; DMR14; Fra+16]
Some Motivations

- $\Gamma \vdash \ ? : \sigma$

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- Could we think of intersection types as a kind of *finite-dimensional* descriptions of terms?
Some Motivations

- $\Gamma \vdash ? : \sigma$
  Given an environment $\Gamma$ and a type $\sigma$, is there a term $M$ such that $\Gamma \vdash M : \sigma$?

- Constructively, this is a program *synthesis* problem. Intersection types have been proposed as an important foundation for type theoretic synthesis [Düd+12; DMR14; Fra+16].

- Could we think of intersection types as a kind of *finite-dimensional* descriptions of terms?

Main Results

- Proof theoretic notions of *norm* and *dimension* for intersection types.
- Dimension is a bounding principle orthogonal to the notion of *rank*.
- Inhabitation under dimensional bound is \textsc{ExpSpace}-complete and subsumes (infinitely) decidability in rank 2.
Intersection Types

Five-set Venn diagram using congruent ellipses in a 5-fold rotationally symmetrical arrangement devised by Branko Grünbaum.
https://commons.wikimedia.org/w/index.php?curid=14250677
Strict Intersection Type System

Definition (λ-Terms)

\[ M, N ::= x \mid (\lambda x. M) \mid (M \ N) \]

Definition (Strict Intersection Types)

\[ A, B ::= a \mid \sigma \rightarrow A \]
\[ \sigma, \tau ::= A_1 \cap \cdots \cap A_n \quad n \geq 1 \]

Definition (Strict Type Assignment [Bak11](Def. 5.1))

\(\frac{1 \leq i \leq n}{\Gamma, x : \bigcap_{i=1}^{n} A_i \vdash_S x : A_i} \) \quad (\text{Var}) \quad \frac{\Gamma \vdash_S M : A_i \text{ for } i = 1 \ldots n}{\Gamma \vdash_S M : \bigcap_{i=1}^{n} A_i} \quad (\text{∩I})

\(\frac{\Gamma \vdash_S M : \sigma \rightarrow A \quad \Gamma \vdash_S N : \sigma}{\Gamma \vdash_S M \ N : A} \) \quad (\rightarrow \text{E}) \quad \frac{\Gamma, x : \sigma \vdash_S M : A}{\Gamma \vdash_S \lambda x. M : \sigma \rightarrow A} \quad (\rightarrow \text{I})
Inhabitation

\[ \vdash ? : (a \to c) \to (b \to a \to c) \to a \to b \to c \]
Inhabitation

\[ \vdash ? : (a \to c) \to (b \to a \to c) \to a \to b \to c \]

\[ \{ f : a \to c, g : b \to a \to c, x : a, y : b \} \vdash X : c \]
Inhabitation

\[ \vdash \exists f : a \to c, g : b \to a \to c, x : a, y : b \vdash x : c \]

\[ \vdash \exists f \forall \vdash f \exists \]

\[ \vdash \exists g \forall \vdash g \exists \]
Inhabitation

\[ \vdash \exists : (a \to c) \to (b \to a \to c) \to a \to b \to c \]

\[ \{ f : a \to c, g : b \to a \to c, x : a, y : b \} \vdash x : c \]

\[ \exists \]

\[ \exists \]

\[ \{ \ldots \} \vdash f \cdot y : c \]

\[ \{ \ldots \} \vdash g \cdot z \cdot w : c \]

\[ \{ \ldots \} \vdash y : a \]
\[ \vdash \ ? : (a \to c) \to (b \to a \to c) \to a \to b \to c \]

\[ \{\ldots\} \vdash \lambda f. \lambda g. \lambda x. \lambda y. fx : \sigma \]

\[ \{f : a \to c, g : b \to a \to c, x : a, y : b\} \vdash x : c \]

\[ \{\ldots\} \vdash fx : c \]

\[ \{\ldots\} \vdash fY : c \]

\[ \{\ldots\} \vdash gZW : c \]

\[ \{\ldots\} \vdash Y : a \]

\[ \{\ldots\} \vdash x : a \]
Inhabitation

\[ \vdash ? : (a \to c) \to (b \to a \to c) \to a \to b \to c \]

\[ \{ \ldots \} \vdash \lambda f. \lambda g. \lambda x. \lambda y. fx : \sigma \]

\[ \{ f : a \to c, g : b \to a \to c, x : a, y : b \} \vdash \mathcal{X} : c \]

\[ \{ \ldots \} \vdash fx : c \]

\[ \{ \ldots \} \vdash fY : c \]

\[ \{ \ldots \} \vdash W : a \]

\[ \{ \ldots \} \vdash Z : b \]

\[ \{ \ldots \} \vdash Y : a \]

\[ \{ \ldots \} \vdash x : a \]
Inhabitation

\[
\vdash \exists f : (a \to c) \to (b \to a \to c) \to a \to b \to c
\]

\[
\{\ldots\} \vdash \lambda f.\lambda g.\lambda x.\lambda y. f x : \sigma
\]

\[
\{\ldots\} \vdash \lambda f.\lambda g.\lambda x.\lambda y. g y x : \sigma
\]

\[
\{\{f : a \to c, g : b \to a \to c, x : a, y : b\} \vdash \exists \lambda x.\lambda y. f x : c
\]

\[
\{\{f x : c\} \vdash f y : c
\]

\[
\{\{f y : c\} \vdash f x : c
\]

\[
\{\ldots\} \vdash f x : c
\]

\[
\{\ldots\} \vdash f y : c
\]

\[
\{\ldots\} \vdash f x : c
\]

\[
\{\ldots\} \vdash \exists f.\lambda g.\lambda x.\lambda y. g y x : \sigma
\]

\[
\{\ldots\} \vdash \lambda f.\lambda g.\lambda x.\lambda y. f x : \sigma
\]

\[
\{\ldots\} \vdash \lambda f.\lambda g.\lambda x.\lambda y. y x : \sigma
\]

\[
\{\ldots\} \vdash g Z W : c
\]

\[
\{\ldots\} \vdash g y x : c
\]

\[
\{\ldots\} \vdash g W : a
\]

\[
\{\ldots\} \vdash g y x : c
\]

\[
\{\ldots\} \vdash W : a
\]

\[
\{\ldots\} \vdash g y x : c
\]

\[
\{\ldots\} \vdash y : b
\]

\[
\{\ldots\} \vdash y : b
\]

\[
\{\ldots\} \vdash x : a
\]

\[
\{\ldots\} \vdash x : a
\]
Inhabitation

\[ \vdash ? : (a \rightarrow c) \rightarrow (b \rightarrow a \rightarrow c) \rightarrow a \rightarrow b \rightarrow c \]

\[ \{\ldots\} \vdash \lambda f. \lambda g. \lambda x. \lambda y. f x : \sigma \]

\[ \{\ldots\} \vdash \lambda f. \lambda g. \lambda x. \lambda y. g y x : \sigma \]

\[ \{\ldots\} \vdash f : a \rightarrow c, g : b \rightarrow a \rightarrow c, x : a, y : b \vdash \mathcal{X} : c \]

\[ \{\ldots\} \vdash f x : c \]

\[ \{\ldots\} \vdash f \mathcal{Y} : c \]

\[ \{\ldots\} \vdash q \mathcal{Z} W : c \]

\[ \{\ldots\} \vdash g y x : c \]

\[ \{\ldots\} \vdash \mathcal{Y} : a \]

\[ \{\ldots\} \vdash x : a \]

\[ \{\ldots\} \vdash \mathcal{Z} : b \]

\[ \{\ldots\} \vdash y : b \]

\[ \{\ldots\} \vdash W : a \]

\[ \{\ldots\} \vdash x : a \]

- For simple types this is APTIME = PSPACE, and inhabitation is PSPACE-complete (Statman 1979)
Inhabitation

\{ f : ((a \rightarrow a) \rightarrow c) \land ((b \rightarrow b) \rightarrow d) \} \vdash ? : c \land d
Inhabitation

\[ \{ f : ((a \rightarrow a) \rightarrow c) \cap ((b \rightarrow b) \rightarrow d) \} \vdash ? : c \cap d \]
\[ \{ f : ((a \rightarrow a) \rightarrow c) \cap ((b \rightarrow b) \rightarrow d) \} \vdash ? : c \cap d \]
Inhabitation

\[ \{ f : ((a \to a) \to c) \cap ((b \to b) \to d) \} \vdash ? : c \cap d \]

\[ \forall \quad \forall \]

\[ \{ \ldots \} \vdash \mathcal{X} : c \]

\[ \{ \ldots \} \vdash \mathcal{Y} : a \to a \]

\[ \{ \ldots, z : a \} \vdash \mathcal{Z} : a \]

\[ \{ \ldots, z : b \} \vdash \mathcal{Z} : b \]

\[ \{ \ldots \} \vdash \mathcal{X} : a \]

\[ \{ \ldots \} \vdash \mathcal{Y} : b \to b \]
\{ f : ((a \rightarrow a) \rightarrow c) \cap ((b \rightarrow b) \rightarrow d) \} \vdash ? : c \cap d

\begin{align*}
\{\ldots\} & \vdash f \lambda z. z : c \\
\{\ldots\} & \vdash \mathcal{X} : c \\
\{\ldots\} & \vdash \lambda z. z : a \rightarrow a \\
\{\ldots, z : a\} & \vdash z : a \\
\{\ldots, z : a\} & \vdash Z : a \\
\{\ldots\} & \vdash \lambda z. z : a \rightarrow b \\
\{\ldots, z : b\} & \vdash Z : b \\
\{\ldots, z : b\} & \vdash z : b
\end{align*}
Inhabitation

\[ \{ f : ((a \to a) \to c) \cap ((b \to b) \to d) \} \vdash ? : c \cap d \]

- Simultaneous constraints are *coupled* through \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \)
Urzyczyn’s Analysis [Urz09]

Rank-Based Analysis by Paweł Urzyczyn

Definition (⊢_S? : σ)

Given a type σ is there a λ-term M such that ⊢_S M : σ?

Definition (Rank [Lei83])

\[
\begin{align*}
\text{rank}(\tau) &= 0 \text{ if } \tau \text{ is a simple type} \\
\text{rank}(\sigma \to \tau) &= \max(\text{rank}(\sigma) + 1, \text{rank}(\tau)) \\
\text{rank}(\sigma \cap \tau) &= \max(1, \text{rank}(\sigma), \text{rank}(\tau))
\end{align*}
\]

- The inhabitation problem is undecidable [Urz99]
- ⊢_S? : σ with rank(σ) ≤ 2 is EXPSPACE-complete [Urz09]
- ⊢_S? : σ with rank(σ) ≥ 3 is undecidable [Urz09]
Rank-Based Analysis by Paweł Urzyczyn

\( \{ f : ((a \to a) \to c) \cap ((b \to b) \to d) \} \vdash ? : c \cap d \)

- Linear bounded tree width
- EXPSPACE-complete model of “bus machines”
Beyond rank 2?

- There is a lot of room in exponential space!
Beyond rank 2?

- There is a lot of room in exponential space!
- But the jump from rank 2 to rank 3 is *infinite*
Beyond rank 2?

- There is a lot of room in exponential space!
- But the jump from rank 2 to rank 3 is *infinite*
- We show that dimensional bound provides a rank-independent stratification, decidable in each dimension, and subsuming rank 2 in Expspace
Beyond rank 2?

- There is a lot of room in exponential space!
- But the jump from rank 2 to rank 3 is *infinite*
- We show that dimensional bound provides a rank-independent stratification, decidable in each dimension, and subsuming rank 2 in \textsc{Expspace}
- ... and it has a logical meaning
Elaboration Systems

Engineering drawing from da Vinci's journals
Set Theoretic Elaborations

**Definition (Elaborations)**

\[ P, Q, R ::= x[S] | (\lambda x. P)[S] | (P \ Q)[S] \]  
where \( S = \{A_1, \ldots, A_n\}, n \geq 1 \)

**Definition (\( P \sqcup Q \), defined for [\( P \equiv Q \)] )**

\[
\begin{align*}
  x[S] \sqcup x[S'] &\equiv x[S \cup S'] \\
(\lambda x. P)[S] \sqcup (\lambda x. Q)[S'] &\equiv (\lambda x. P \sqcup Q)[S \cup S'] \\
(PQ)[S] \sqcup (P'Q')[S'] &\equiv ((P \sqcup P')(Q \sqcup Q'))[S \cup S']
\end{align*}
\]

**Definition (Norm \( ||\cdot|| \))**

\[
\begin{align*}
  ||x[S]|| &= |S| \\
  ||(\lambda x. P)[S]|| &= \max\{||P||, |S|\} \\
  ||(PQ)[S]|| &= \max\{||P||, ||Q||, |S|\}
\end{align*}
\]
### Set Theoretic Elaborations

**Definition (Elaborations)**

\[ P, Q, R ::= x[S] \mid (\lambda x. P)[S] \mid (P \ Q)[S] \text{ where } S = \{A_1, \ldots, A_n\}, \ n \geq 1 \]

**Definition (\(P \sqcup Q\), defined for \([P] \equiv [Q]\))**

\[
x[S] \sqcup x[S'] \equiv x[S \cup S'] \\
(\lambda x. P)[S] \sqcup (\lambda x. Q)[S'] \equiv (\lambda x. P \sqcup Q)[S \cup S'] \\
(PQ)[S] \sqcup (P'Q')[S'] \equiv ((P \sqcup P')(Q \sqcup Q'))[S \cup S']
\]

**Definition (Norm \(|\cdot|\))**

\[
|\|x[S]\|| = |S| \\
|\|(\lambda x. P)[S]\|| = \max\{|\|P\||, |S|\} \\
|\|(PQ)[S]\|| = \max\{|\|P\||, |\|Q\||, |S|\}
\]

Non-negativity : \(|\|P\|| > 0\)
Subadditivity : \(|\|P \sqcup Q\|| \leq |\|P\|| + |\|Q\||\) for \([P] \equiv [Q]\)
Set Theoretic Elaboration System

Definition \((\Gamma \vdash M \mapsto P : \sigma)\)

\[
\frac{1 \leq i \leq n}{\Gamma, x : \bigcap_{i=1}^{n} A_i \vdash x \mapsto x[A_i] : A_i} \quad \text{(Var)}
\]

\[
\frac{\Gamma, x : \sigma \vdash M \mapsto P : A}{\Gamma \vdash \lambda x. M \mapsto (\lambda x. P)[\sigma \rightarrow A] : \sigma \rightarrow A} \quad \text{\rightarrow I)}
\]

\[
\frac{\Gamma \vdash M \mapsto P : \sigma \rightarrow A \quad \Gamma \vdash N \mapsto Q : \sigma}{\Gamma \vdash MN \mapsto (P Q)[A] : A} \quad \text{\rightarrow E)}
\]

\[
\frac{\Gamma \vdash M \mapsto P_i : A_i \text{ for } i = 1 \ldots n}{\Gamma \vdash M \mapsto \sqcap_{i=1}^{n} P_i : \sqcap_{i=1}^{n} A_i} \quad \text{(\sqcap I)}
\]

Intuition
The operation \(\sqcap\) allows us to measure usage of \((\cap I)\) as a logical resource under norm \(\| \cdot \|\).
Set Theoretic Elaboration System

Definition ($\Gamma \vdash M \rightarrowto P : \sigma$)

\[
\begin{align*}
1 \leq i \leq n & \quad \Gamma, x : \bigcap_{i=1}^{n} A_i \vdash x[A_i] : A_i \quad \text{(Var)} \\
\Gamma, x : \sigma \vdash M \rightarrowto P : A & \quad \Gamma \vdash \lambda x.M \rightarrowto (\lambda x.P)[\sigma \rightarrow A] : \sigma \rightarrow A \quad \text{($\rightarrow$I)} \\
\Gamma \vdash M \rightarrowto P : \sigma \rightarrow A & \quad \Gamma \vdash N \rightarrowto Q : \sigma \quad \Gamma \vdash M N \rightarrowto (P Q)[A] : A \quad \text{($\rightarrow$E)} \\
\Gamma \vdash M \rightarrowto P_i : A_i \text{ for } i = 1 \ldots n & \quad \Gamma \vdash M \rightarrowto \bigsqcup_{i=1}^{n} P_i : \bigcap_{i=1}^{n} A_i \quad \text{($\cap$I)}
\end{align*}
\]

Intuition

The operation $\bigsqcup_{i=1}^{n} P_i$ allows us to measure usage of ($\cap$I) as a logical resource under norm $\|\bullet\|$
Dimension
Definition ($\lambda_n^{[\cap]}$)

$\Gamma \vdash_n M : \sigma$ iff $\exists P. \Gamma \vdash M \rightarrow P : \sigma$ with $\|P\| \leq n$

Lemma

$\Gamma \vdash S M : \sigma$ iff $\Gamma \vdash_n M : \sigma$ for some $n > 0$

Definition (Dimension)

The set theoretic dimension of a term $M$ at $\Gamma$ and $\sigma$ is

$$\dim^\sigma_\Gamma = \min\{n \mid \Gamma \vdash_n M : \sigma\}$$
Subject Reduction

Marilyn Diptych
http://www.tate.org.uk/servlet/ViewWork?workid=15976&tabview=image,
Subject Reduction in Bounded Dimension

Terms can be elaborated in non-increasing norm under $\beta$-reduction:

**Theorem (Subject Reduction for $\lambda^{[n]}_n$)**

If $\Gamma \vdash M \rightarrow P : \tau$ and $M \rightarrow_{\beta} M'$, then there exists $R$ with $\|R\| \leq \|P\|$ such that $\Gamma \vdash M' \rightarrow R : \tau$

Consequences:

- Each dimensional fragment $\lambda^{[n]}_n$ is a meaningful type system.
- Inhabitation in bounded dimension for $\lambda^{[n]}_n$ can be limited to search for normal forms.
Inhabitation in Bounded Set Theoretic Dimension

Problem (Inhabitation for $\lambda^{[n]}$)

Given environment $\Gamma$, type $\sigma$ and number $n > 0$:
is there a term $M$ such that $\Gamma \vdash_n M : \sigma$?
Inhabitation in Bounded Set Theoretic Dimension

Problem (Inhabitation for $\lambda^{[n]}$)

Given environment $\Gamma$, type $\sigma$ and number $n > 0$:
is there a term $M$ such that $\Gamma \vdash_n M : \sigma$?

Lemma (Subformula property, [BCDC83] Lemma 4.5)

If $\Gamma \vdash N \leftrightarrow P : \sigma$ where $N$ is in normal form,
then there exists $Q$ containing only subformulae of $\sigma$ and types appearing
in $\Gamma$ such that

\[ \Gamma \vdash N \leftrightarrow Q : \sigma \]

Theorem

The inhabitation problem for $\lambda^{[n]}$ is undecidable.

Proposition

For $n = 1$ set-theoretic inhabitation is $\mathsf{PSPACE}$-complete ([RU12] Cor. 22).
Beyond Rank 2 Inhabitation

Camille Flammarions 1833 (excerpt)
https://commons.wikimedia.org
## Multiset Elaborations

### Definition (Multiset Elaborations)

\[ \varphi, \psi ::= a \mid s \rightarrow \varphi \quad s, t ::= \langle \varphi_1, \ldots, \varphi_n \rangle \quad (n > 0) \]

\[ P, Q, R ::= x(s) \mid (\lambda x.P)(s) \mid (P \ Q)(s) \]

### Definition (\( P \sqsubseteq Q \))

\[ x(s) \sqsubseteq x(s') \equiv x(s \sqcup s') \]

\[ (\lambda x.P)(s) \sqsubseteq (\lambda x.Q)(s') \equiv (\lambda x.P \sqsubseteq Q)(s \sqcup s') \]

\[ (PQ)(s) \sqsubseteq (P'Q')(s') \equiv ((P \sqsubseteq P')(Q \sqsubseteq Q'))(s \sqcup s') \]

### Definition (\( \|P\| \))

\[ \|x(s)\| = |s| \]

\[ \|(\lambda x.P)(s)\| = \max\{\|P\|, |s|\} \]

\[ \|(P \ Q)(s)\| = \max\{\|P\|, \|Q\|, |s|\} \]
Multiset Elaboration System

Definition \((\Delta \vdash M \Rightarrow P : s)\)

\[
1 \leq i \leq n \\
\frac{\Delta, x : \langle \varphi_1, \ldots, \varphi_n \rangle \vdash x \Rightarrow x\langle \varphi_i \rangle : \langle \varphi_i \rangle}{(\text{Var})}
\]

\[
\frac{\Delta, x : s \vdash M \Rightarrow P : \langle \varphi \rangle}{\Delta \vdash \lambda x. M \Rightarrow (\lambda x. P)(s \rightarrow \varphi) : \langle s \rightarrow \varphi \rangle} \quad (\rightarrow I)
\]

\[
\frac{\Delta \vdash M \Rightarrow P : \langle s \rightarrow \varphi \rangle \quad \Delta \vdash N \Rightarrow Q : s}{\Delta \vdash MN \Rightarrow (P \ Q)(\varphi) : \langle \varphi \rangle} \quad (\rightarrow E)
\]

\[
\frac{\Delta \vdash M \Rightarrow P_i : \langle \varphi_i \rangle \text{ for } i = 1 \ldots n}{\Delta \vdash M \Rightarrow \bigoplus_{i=1}^n P_i : \langle \varphi_1, \ldots, \varphi_n \rangle} \quad (\text{\textbullet})
\]

where \((\text{\textbullet})\) is for each \(x\langle s \rangle\) in \(\bigoplus_{i=1}^n P_i\) if \(x\) free in \(M\), then \(s \subseteq \Delta(x)\).
Multiset Elaboration System

**Definition** \((\Delta \vdash M \implies P : s)\)

\[
\begin{align*}
1 \leq i \leq n & \\
\Delta, x : \langle \varphi_1, \ldots, \varphi_n \rangle \vdash x \implies x\langle \varphi_i \rangle : \langle \varphi_i \rangle \\
\Delta, x : s \vdash M \implies P : \langle \varphi \rangle & \\
\Delta \vdash \lambda x. M \implies (\lambda x.P)\langle s \rightarrow \varphi \rangle : \langle s \rightarrow \varphi \rangle \\
\Delta \vdash M \implies P : \langle s \rightarrow \varphi \rangle & \quad \Delta \vdash N \implies Q : s \\
\Delta \vdash M \cdot N \implies (P \cdot Q)\langle \varphi \rangle : \langle \varphi \rangle & \\
\Delta \vdash M \implies P_i : \langle \varphi_i \rangle & \text{for } i = 1 \ldots n \\
\Delta \vdash M \implies \bigoplus_{i=1}^n P_i : \langle \varphi_1, \ldots, \varphi_n \rangle & \\
\end{align*}
\]

where \((\star)\) is for each \(x\langle s \rangle\) in \(\bigoplus_{i=1}^n P_i\) if \(x\) free in \(M\), then \(s \subseteq \Delta(x)\).

**Non-linearity, for example** \(\lambda x. (x \ x)\):

\[\vdash (\lambda x. (x \langle a \rightarrow a \rangle \ x\langle a \rangle)((a \cap (a \rightarrow a))) \rightarrow a) : \langle (a \cap (a \rightarrow a)) \rightarrow a \rangle\]
Multiset Elaborated System

Definition (Set-theoretic Collapse)

\[ a^\circ \equiv a \]
\[ (s \to \varphi)^\circ \equiv s^\circ \to \varphi^\circ \]
\[ \langle \varphi_1, \ldots, \varphi_n \rangle^\circ \equiv \varphi_1^\circ \cap \cdots \cap \varphi_n^\circ \]

Definition (\(\lambda^{(n)}_n\))

\(\Gamma \vdash_n M : \sigma\) iff

\(\exists \Delta, \mathbb{P}, s. (\Delta \vdash M \Rightarrow \mathbb{P} : s\) with \(\Gamma = \Delta^\circ\) and \(\sigma = s^\circ\) and \(||\mathbb{P}|| \leq n)\)

\(\lambda^{(n)}_n\) is defined in terms of standard (idempotent) intersection types and a proof-theoretic measure.
Theorem (Subject Reduction)

If $\Delta \vdash M \iff P : t$ and $M \beta \rightarrow M'$,
then there exists $R$ with $\|R\| \leq \|P\|$ such that

$\Delta \vdash M' \iff R : t$
Theorem (Subject Reduction)

If $\Delta \vdash M \iff P : t$ and $M \rightarrow_\beta M'$,
then there exists $R$ with $\|R\| \leq \|P\|$ such that
$\Delta \vdash M' \iff R : t$

Theorem

The inhabitation problem $\Gamma \vDash_n ? : \sigma$ in bounded multiset dimension is EXPSPACE-complete.
For each dimensional bound $d > 0$, inhabitation is in ATIME($N^{2d}$) where $N$ denotes the size of the input $\Gamma$ and $\sigma$. 
Theorem (Subject Reduction)

If $\Delta \vdash M \iff P : t$ and $M \rightarrow_\beta M'$, then there exists $R$ with $\|R\| \leq \|P\|$ such that $\Delta \vdash M' \iff R : t$

Theorem

The inhabitation problem $\Gamma \vdash n? : \sigma$ in bounded multiset dimension is EXPSPACE-complete. For each dimensional bound $d > 0$, inhabitation is in ATIME($N^{2d}$) where $N$ denotes the size of the input $\Gamma$ and $\sigma$.

Corollary

For each fixed $d$ inhabitation in multiset dimension $d$ is PSPACE-complete.
Proposition

Suppose we can derive the judgement $\Delta \vdash N \iff P : \langle \varphi_1, \ldots, \varphi_n \rangle$ in rank 2, where $N$ is a normal form. Then $\|P\| = n$.

Substantial generalization of inhabitation in rank 2 fragment.
Multiset Dimension and Parallel Constraints
\[ \Gamma = \{ x : (e \cap f) \to g, y : (((a \to a) \cap (b \to b)) \to e) \cap (((b \to b) \cap (c \to c)) \to f) \} \]
\[ \Gamma \vdash_n ? : g \]

\[ \mathcal{X} \equiv x(y(\lambda z.z)) \]
\[ \mathcal{Y} \equiv y(\lambda z.z) \]

\[ \mathcal{Z} \equiv \lambda z.z \]
\[ \mathcal{W} \equiv z \]

\[ \Gamma \vdash \mathcal{Z} : a \to a \]
\[ \Gamma \vdash \mathcal{Z} : b \to b \]
\[ \Gamma \vdash \mathcal{Z} : b \to b \]
\[ \Gamma \vdash \mathcal{Z} : c \to c \]

\[ \Gamma, z : a \vdash \mathcal{W} : a \]
\[ \Gamma, z : b \vdash \mathcal{W} : b \]
\[ \Gamma, z : b \vdash \mathcal{W} : b \]
\[ \Gamma, z : c \vdash \mathcal{W} : c \]

\[ \| \cdot \| \]
\[ \vdash_n \Rightarrow \| \cdot \| \leq n \]
The World

Cellarius
Harmonia Macrocosmica
Planisphaerium Copernicanum
The World
Future Work

- Decidability of Typability?
  - ✔ **Decidability in both systems and $P_{SPACE}$-complete in set theoretic system** [D & R: *Typability in Bounded Dimension*, Submitted]

- Universal type $\omega$ (empty intersection) implications

- Algebraic perspective
  - proofs as finite vectors
  - $\omega$-elaborated proof as zero
  - $(\cap I)$ as proof addition
  - inversion of $(\cap I)$ as proof subtraction

- Expressiveness of individual fragments

- Reduction complexity

- Model theory
Bibliography I


